Solutions to USC's 2005 High School Math Contest

Notation. The notation $a|b$ will mean that a and b are positive integers and a divides b. For example, $31|2^{10}-1$, since $2^{10}-1 = 1023 = (31)(33)$.

1. **(b)** The profit per apple is \$ $2/5 - 1/3 =$ \$ $1/15$. So, to make a profit of \$10 Tim must buy (and sell) 150 apples.

2. **(d)** The maximal value of the function $\sin x$ is 1 and its minimal value is -1. Since $f(x)$ is continuous, its range is $[\sqrt{9}, \sqrt{25}] = [3, 5]$.

3. **(c)** The exponent of x is $\left(\frac{3}{2} - \frac{2}{3}\right)\frac{3}{5} = \frac{1}{2}$.

4. **(c)** We get $x = y \neq 0$, thus $\frac{7x-y}{x+y} = 3$.

5. **(d)** We have (1) $CD + DA = 14$ and from the Pythagorean theorem $CD^2 + BD^2 = 169$ and $DA^2 + BD^2 = 225$. So, $DA^2 - CD^2 = 56$. Using (1) we get $DA - CD = 4$, and $DA = 9$, $CD = 5$. Therefore $BD = 12$.

6. **(b)**The last three digits of 2005^{2005} are the same as the last three digits of 5^{2005}. Also, the last three digits of the powers of 5 are $5, 25, 125, 625, 125, 625, 125\ldots$, so they start repeating. Thus, the last three digits of 5^{2005} are 125. (More rigorously, 1000 divides $5^{k+2} - 5^k = 24 \times 5^k = 3 \times 1000 \times 5^{k-3}$ when $k \geq 3$.)

7. **(c)** From the second equation we get that either $x = 0$, or $y = \pm 1$. Substituting $x = 0$ in the first equation we get the solutions $(0, 1)$ and $(0, -1)$. Next, when $y = 1$ we get the solutions $(0, 1)$ and $(2, 1)$. Finally, when $y = -1$ we get the solutions $(0, -1)$, $(-2, -1)$. So, the system has four distinct solutions - $(0, 1), (0, -1), (2, 1)$, and $(-2, -1)$.

8. **(e)** Since $a + 2b$ divides 10 and $a + 2b \geq 3$ we get that $a + 2b$ is either 5 or 10. There are two cases. Case I: $a + 2b = 5$ and $a - b = 2$. Then $a = 3$, $b = 1$, and $2a - b = 5$. Case II: $a + 2b = 10$ and $a - b = 1$. Then $a = 4$, $b = 3$, and again $2a - b = 5$. So, in all cases $2a - b = 5$.

9. **(c)** Note that $40000 = 4 \cdot 10^4 = 2^6 5^4$. Since $mn = 2^6 5^4$ and neither m nor n is divisible by 10 we get that one of the numbers m and n equals 2^6 and the other - 5^4. So, $m + n = 2^6 + 5^4 = 689$.

10. **(d)** Since $(\log_x y)(\log_y x) = \log_x x = 1$, we have $(\log_x y)^2 + (\log_y x)^2 = (\log_x y + \log_y x)^2 - 2 = 47$.

11. **(a)** Since $\left(\frac{3}{2}\right)^2 > 2$ and $\left(\frac{3}{2}\right)^3 > 3$, $\sqrt{2}$ and $\sqrt[3]{3}$ are smaller than $\frac{3}{2}$. Also, $3^2 > 2^3$, so $\log_2 3 > \frac{3}{2}$, and $5^2 < 3^3$, implying $\log_3 5 < \frac{3}{2}$.

12. **(a)** Denote by k the distance which the slower particle travels in 50 seconds. Then $50r = k$ and $50R = k+300$. Subtracting the above equations we get $50(R-r) = 300$ or $R - r = 6$ ft./sec.

13. **(e)** If $m = 1$ the lines are parallel. If $m \neq 1$ they intersect at the point $\left(\frac{5}{1-m}, \frac{2m+3}{1-m}\right)$. Since both coordinates of the point of intersection must be positive, we get $\frac{5}{1-m} > 0$, so $m < 1$. Also, $\frac{2m+3}{1-m} > 0$, so $m > -3/2$.

14. **(a)** Substituting $x = \frac{\pi}{2}$ and then $x = -\frac{\pi}{2}$ in the equation we get $f\left(\frac{\pi}{2}\right)+2f\left(-\frac{\pi}{2}\right) = 1$ and $f\left(-\frac{\pi}{2}\right) + 2f\left(\frac{\pi}{2}\right) = -1$. Solving the last two equations for $f\left(\frac{\pi}{2}\right)$ we get $f\left(\frac{\pi}{2}\right) = -1$.

15. **(b)** First, we show that one can fit three non-intersecting circles of radius 1 inside the circle of radius 2.4 (we call it the "big circle"). Denote by O the center of the "big" circle and let A, B, C be three points such that $\triangle ABC$ is an equilateral triangle with O - the center of its circumscribed circle, and with $OA = OB = OC = 1.3$. Clearly, $\angle AOB = \angle BOC = \angle COA = 120°$. By the Law of Cosines, $AB^2 = OA^2 + OB^2 - 2OA \cdot OB\cos(120°) = 1.3^2 + 1.3^2 + 1.3^2 = 5.07 > 4$. So, $AB = AC = BC > 2$. Consider the three circles of radius 1 (we call them "little circles") whose centers are the points A, B, and C. The little circles do not intersect since the distances between any two centers of distinct little circles is > 2. Also, the three little circles are inside the big circle since the distance from O to any of the centers of the little circles is < 1.4.

    Next, we show that one cannot fit four non-intersecting little circles inside the big circle. Indeed, let A, B, C, and D be the centers of four little circles inside the big circle. Then OA, OB, OC, and OD are all < 1.4. If the points are labeled in a suitable way, we have $\angle AOB + \angle BOC + \angle COD + \angle DOA = 360°$ (if not, we relabel the points). Then, at least one of the angles $\angle AOB, \angle BOC, \angle COD, \angle DOA$ is $\leq 90°$. Suppose $\angle AOB \leq 90°$. Again, by the Law of Cosines we get $AB^2 = OA^2 + OB^2 - 2OA \cdot OB\cos(\angle AOB) \leq OA^2 + OB^2 < 1.4^2 + 1.4^2 = 3.92 < 4$. Thus, $AB < 2$ and the circles whose centers are A and B intersect. We proceed similarly if $\angle BOC, \angle COD$, or $\angle DOA$ is $\leq 90°$.

16. **(a)** Let D be the point on the straight line AB such that CD is perpendicular to AB. Then CD is the height of $\triangle ABC$. Since the base of $\triangle ABC$ is $\sqrt{3}-1$ and its area is $\frac{\sqrt{3}-1}{2}$, we get $CD = 1$. Now, considering the right triangle $\triangle ADC$ we obtain $AD = \sqrt{3}$ and $\angle CAD = 30°$. Also, since $AD = \sqrt{3}$ and $AB = \sqrt{3} - 1$, we get $BD = 1$. Finally, considering the right triangle $\triangle BDC$ we obtain $\angle BCD = 45°$. So, $\angle ACB = 15°$.

17. **(b)** Let $N = 1234 \cdots 424344$. Then N is divisible by 9 since $1+2+3+\cdots+41+42+43+44 = 990$ is divisible by 9. Also, the remainder when N is divided by 5 is 4. The only integer between 0 and 44 which is divisible by 9 and has remainder 4 when divided by 5 is 9.

18. **(c)** Since $\sin(90° - x) = \cos x$ and $\cos(90° - x) = \sin x$ we get $\tan(90° - x)\tan x = 1$. Applying the above formula for $x = 5°, 15°, 25°$, and $35°$, and taking into account $\tan 45° = 1$, we get that the value of the product is 1.

19. **(b)** If P is a point on the x-axis,

$$PA + PB = PA + PB' \geq AB' = 13,$$

where $B' = (6.5, -3)$ (the reflection of B with respect to the x-axis), and equality holds when P is the point of intersection of the straight line AB' and the x-axis.

Similarly, if P is a point on the y-axis,

$$PA + PB = PA + PB'' \geq AB'' = 10,$$

where $B'' = (-6.5, 3)$ (the reflection of B with respect to the y-axis), and equality holds when P is the point of intersection of the straight line AB'' and the y-axis.

20. **(c)** One can paint five cars using 3 colors in $3^5 = 243$ ways. Also, one can paint five cars using only 2 out of 3 colors in $3 \cdot 2^5 = 96$ ways (there are 3 ways to pick 2 out of 3 colors). Finally, one can paint five cars using only 1 out of 3 colors in $3 \cdot 1^5 = 3$ ways. By the inclusion-exclusion principle, the answer is $243 - 96 + 3 = 150$.

21. **(b)** Let d be the greatest common divisor of a and b. Then $6d^2 | 36ma^2 - 6nb^2 = 1008$, so $d^2 | 168$. Since $168 = 2^3 \cdot 3 \cdot 7$, the only squares of integers which divide 168 are 1 and 4. Thus, $d \leq 2$, and when $a = b = 2$, $m = 7$, $n = 0$ we have $d = 2$.

22. **(d)** The median can not be 1 (there are at least three numbers in the set larger than 1). For similar reasons, the median is not 9. The median could be 6. Then $6 = (x+6+4+1+9)/5$, so $x = 10$. Since 6 is the median of the set $\{1, 4, 6, 9, 10\}$ and the mean of the set is also 6, the number 6 has the described property. Further, if the median is 4, then $4 = (x+6+4+1+9)/5$, so $x = 0$. Again, one checks that $x = 0$ has the desired property. Finally, if the median is x, then $x = (x+6+4+1+9)/5$, so $x = 5$, and one checks that $x = 5$ has the desired property, too.

23. **(d)** Since $x-1|x^n-1$ when x and n are positive integers, then $2^a-1|2^b-1$ whenever a and b are positive integers and $a|b$ (set $x=2^a$ and $n=b/a$). Thus, $3=2^2-1$ divides $2^{1650}-1$ since $2|1650$. Similarly, $7=2^3-1$ divides $2^{1650}-1$ ($3|1650$). In the same way, $31=2^5-1$ divides $2^{1650}-1$, and $2047=2^{11}-1$ divides $2^{1650}-1$. On the other hand, $127=2^7-1$ does not divide $2^{1650}-1$. (Indeed, $7|1652$ so $127|2^{1652}-1$. Thus $127|2^{1650}-1$ would imply $127|(2^{1652}-1)-4(2^{1650}-1)$ or $127|3$.)

24. **(b)** The polynomial $Q(x)$ must be a quadratic polynomial with leading coefficient 1 or -1. We can assume that the leading coefficient of $Q(x)$ is 1 (if not, we can work with $-Q(x)$ since $(Q(x))^2=(-Q(x))^2$). So, $Q(x)=x^2+cx+d$ and

$$(Q(x))^2=x^4+(2c)x^3+(c^2+2d)x^2+(2cd)x+d^2=x^4+2x^3-x^2+ax+b.$$

So, $2c=2$, $c^2+2d=-1$, $2cd=a$, and $d^2=b$. We get $c=1$, $d=-1$, $a=-2$, $b=1$, and $a+b=-1$.

25. **(c)** Suppose B has coordinates (p,q) and C - (r,s). Then

(1) $\quad p>0, r>0, \quad p^2-q^2=1, \quad r^2-s^2=1,$

and

(2) $\quad AB^2=(p+1)^2+q^2=AC^2=(r+1)^2+s^2=BC^2=(p-r)^2+(q-s)^2.$

From (1) $q^2=p^2-1$ and $s^2=r^2-1$. Substituting in (2) we get $(p+1)^2+p^2-1=(r+1)^2+r^2-1$, or $2(p-r)(p+r+1)=0$ which implies $p=r$. Now, substituting in (1) we obtain $p^2-q^2=p^2-s^2=1$, so $q=-s$. Therefore C is a reflection of B with respect to the x-axis. Referring to (2) again we get $(p+1)^2+p^2-1=(p-p)^2+(q-(-q))^2=4q^2=4(p^2-1)$. Simplifying we obtain $p(p+1)=2(p+1)(p-1)$, so $p=2$, $q=\sqrt{3}$. Let $D=(\sqrt{3},0)$ (the midpoint of BC). Then AD is perpendicular to BC and the area of $\triangle ABC$ is $(BC)(AD)/2=3\sqrt{3}$.

26. **(d)** Clearly, $x\neq 0$ and $y\neq 0$. Solving for y we get

$$y=\frac{4x}{x-4}=4+\frac{16}{x-4}.$$

So, $x-4$ divides 16 and since $x\neq 0$, $y\neq 0$, $x-4$ must be in the set $\{-16,-8,-2,-1,1,2,4,8,16\}$. So, there are nine integer solutions $(-12,3)$, $(-4,2)$, $(2,-4)$, $(3,-12)$, $(5,20)$, $(6,12)$, $(8,8)$, $(12,6)$, and $(20,5)$.

27. **(a)** There $(5)(4)=20$ ways to pick **distinct** kinds of flowers for the north and south plots. Also, if distinct kinds of flowers are planted on plots north and south, there are 3 kinds of flowers left to plant in plots west and east. So, in the case when distinct kinds of flowers are planted on plots north and south, there are a

total of $(20)(3)(3) = 180$ ways to plant flowers in the plots, so that flowers in plots which share a common edge are different.

Next, there are 5 ways to pick **the same** kind of flower for the north and south plots. In this case, there are 4 kinds of flowers left to plant in plots west and east. So, in the case when the same kind of flowers are planted on plots north and south, there are a total of $(5)(4)(4) = 80$ ways to plant flowers in the plots, so that flowers in plots which share a common edge are different. So, the total number of choices is 260.

28. **(e)** Denote Amy's answers by (A), Bob's by (B), Cathy's by (C), Dave's by (D), and Eva's by (E). Assume $v = 2$. Using that each contestant had found the correct value of exactly one unknown and (B), (D) we get $y \neq 3$, $x \neq 4$. Next, (E) implies $y = 1$, so $z \neq 1$ which contradicts (A).

    Therefore, $v \neq 2$.

    Again, using (B) and (D) we get $y = 3$, $x = 4$. Next (C) implies $z = 5$. Since $v \neq 2$, we obtain $v = 1$, $u = 2$.

29. **(a)** Since the two triangles are similar we get, $\dfrac{DC}{DA} = \dfrac{AC}{AB} = \dfrac{AD}{DB}$. Since $AC = 15$, and $AB = 20$, we get $DC = \frac{3}{4}DA$ and $DB = \frac{4}{3}DA$. The last two equations imply $DB - DC = \frac{7}{12}DA$. Now, $DB - DC = CB = 7$, so $DA = 12$ and $DC = 9$.

30. **(e)** We have that $f(x) = 2x - 1$ for $x = 1, 2, 3$. Thus, $f(x) - (2x - 1)$ is a polynomial of degree 4, with leading coefficient 1, and three of its roots are the numbers 1,2, and 3. Therefore

$$(*) \quad f(x) - (2x - 1) = (x - 1)(x - 2)(x - 3)(x - k),$$

    where k is some real number. Substituting $x = 0$, and $x = 4$ in (*) we get $f(0) - (-1) = 6k$ and $f(4) - 7 = 6(4 - k)$. Adding the last two equations we get $f(0) + f(4) = 30$.