

High School Math Contest
University of South Carolina
January 29, 2005

1. Tim buys apples at three for \$1. He resells them at five for \$2. Assuming he resells every apple that he buys, how many apples must Tim buy in order to make a profit of \$10 ?

- (a) 75 (b) 150 (c) 225 (d) 300 (e) 375

2. What is the range of the function $f(x) = \sqrt{8 \sin^3 x + 17}$?

- (a) $[0, \infty)$ (b) $[0, 5]$ (c) $[0, 23]$ (d) $[3, 5]$ (e) $[3, 23]$

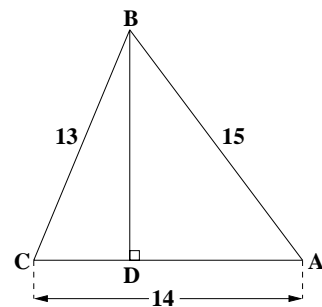
3. For all $x > 0$, $\left(\frac{\sqrt{x^3}}{\sqrt[3]{x^2}}\right)^{3/5} =$

- (a) x^2 (b) x (c) \sqrt{x} (d) $\sqrt[3]{x}$ (e) $\sqrt[4]{x}$

4. Suppose $2x + 3y = 6x - y \neq 0$. Then $\frac{7x - y}{x + y} =$

- (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

5. In $\triangle ABC$, $BC = 13$, $CA = 14$, and $AB = 15$. If D is a point on \overline{CA} such that \overline{BD} is perpendicular to \overline{CA} , then what is BD ?



- (a) 9 (b) 10 (c) 11 (d) 12 (e) 13

6. What are the last three digits of 2005^{2005} ?

- (a) 025 (b) 125 (c) 375 (d) 625 (e) 875

7. How many different real-valued pairs (x, y) satisfy the system of two equations below?

$$\begin{cases} |x - y| = 1 \\ \frac{x}{y} = xy \end{cases}$$

- (a) 2 (b) 3 (c) 4 (d) 5 (e) 6

8. Suppose a and b are positive integers such that $(a + 2b)(a - b) = 10$. What is the value of $2a - b$?

- (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

9. Suppose m and n are positive integers with $mn = 40000$. Suppose further that neither m nor n is divisible by 10. What is the value of $m + n$?

- (a) 650 (b) 660 (c) 689 (d) 691 (e) 695

10. If $\log_y x + \log_x y = 7$, then what is the value of $(\log_y x)^2 + (\log_x y)^2$?

- (a) 40 (b) 43 (c) 45 (d) 47 (e) 49

11. Which of the five numbers below is the largest?

- (a) $\log_2 3$ (b) $\log_3 5$ (c) $\sqrt{2}$ (d) $\sqrt[3]{3}$ (e) $\frac{3}{2}$

12. Two particles move clockwise around a circle with circumference 300 feet. The faster particle moves at a constant speed of R feet per second, and the slower particle moves at a constant speed of r feet per second. If the particles meet every 50 seconds, then what is the value of $R - r$ in feet per second?

(a) 6 (b) 8 (c) 10 (d) 12 (e) 14

13. Let m be a constant. The graphs of the lines $y = x - 2$ and $y = mx + 3$ intersect at a point whose x -coordinate and y -coordinate are both positive if and only if

(a) $m = 1$ (b) $m < 1$ (c) $m > -3/2$
 (d) $-3/2 < m < 0$ (e) $-3/2 < m < 1$

14. Let $f(x)$ be a function such that, for every real number x ,

$$f(x) + 2f(-x) = \sin x.$$

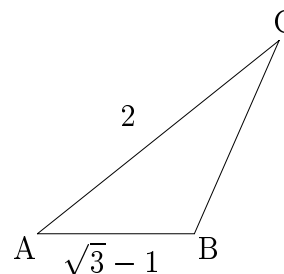
What is the value of $f(\pi/2)$?

(a) -1 (b) $-\frac{1}{2}$ (c) 0 (d) $\frac{1}{2}$ (e) 1

15. What is the largest number of circles of radius 1 that can fit inside a circle of radius 2.4 with no two of the circles of radius 1 having any common points?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

16. Suppose $\triangle ABC$ has area $\frac{\sqrt{3}-1}{2}$, $AB = \sqrt{3}-1$, $AC = 2$, and $\angle CAB$ is acute. What is the measure of $\angle ACB$ (in degrees)?



(a) 15° (b) 18° (c) 20° (d) 22.5° (e) 30°

17. The first 44 positive integers are written in order to form the large number

$$N = 123456789101112 \cdots 424344$$

What is the remainder when N is divided by 45 ?

- (a) 4 (b) 9 (c) 14 (d) 18 (e) 19

18. What is the value of the following product?

$$\tan 5^\circ \tan 15^\circ \tan 25^\circ \tan 35^\circ \tan 45^\circ \tan 55^\circ \tan 65^\circ \tan 75^\circ \tan 85^\circ$$

- (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{3}$ (c) 1 (d) $\sqrt{3}$ (e) 2

19. Let $A = (1.5, 9)$ and $B = (6.5, 3)$ be two points in the plane. If P is a point on the x -axis or the y -axis, then what is the least value for the sum of the distances PA and PB ?

- (a) 9 (b) 10 (c) 11 (d) 12 (e) 13

20. Each car of a five-car train must be painted a solid color. The only color choices are red, blue, and yellow. If each of these colors must be used for at least one car, then how many ways are there to paint this train?

- (a) 100 (b) 125 (c) 150 (d) 175 (e) 200

21. If integers m , n , a , and b satisfy the equation $36ma^2 - 6nb^2 = 1008$, then what is the greatest integer that can divide both a and b ?

- (a) 1 (b) 2 (c) 3 (d) 6 (e) 8

22. What is the number of distinct real numbers x which have the property that the median of the five numbers $x, 6, 4, 1, 9$ is equal to their mean?

- (a) 0 (b) 1 (c) 2 (d) 3 (e) 5

23. Which one of the following integers does not divide $2^{1650} - 1$?
- (a) 3 (b) 7 (c) 31 (d) 127 (e) 2047

24. Let a and b be real numbers such that $x^4 + 2x^3 - x^2 + ax + b = (Q(x))^2$ for some polynomial $Q(x)$. What is the value of $a + b$?
- (a) -2 (b) -1 (c) 0 (d) 1 (e) 2

25. Suppose $\triangle ABC$ is an equilateral triangle with $A = (-1, 0)$ and with both points B and C on the right half of the hyperbola defined by the equation $x^2 - y^2 = 1$. What is the area of $\triangle ABC$?
- (a) $\frac{3\sqrt{3}}{2}$ (b) 3 (c) $3\sqrt{3}$ (d) 6 (e) $4\sqrt{3}$

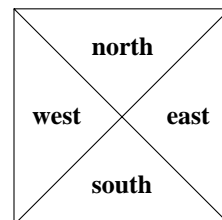
26. A point (x, y) is called integral if both x and y are integers. How many points on the graph of

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$$

are integral points? For example, $(x, y) = (2, -4)$ is one such integral point.

- (a) 6 (b) 7 (c) 8 (d) 9 (e) 10

27. Four triangular gardening plots form a square as in the figure shown. A gardener will plant flowers so that each plot contains one kind of flower, and flowers in plots which share a common edge are different. How many choices does the gardener have if the flowers can be roses, carnations, daisies, lilies, or tulips?



- (a) 260 (b) 270 (c) 280 (d) 290 (e) 300

28. A system of five equations in the unknowns x , y , z , u , and v , has a unique solution which is a permutation of the numbers 1, 2, 3, 4, and 5. Five contestants in a math tournament were asked to solve this system, but they were only able to provide the following partial answers before the end of the allotted time:

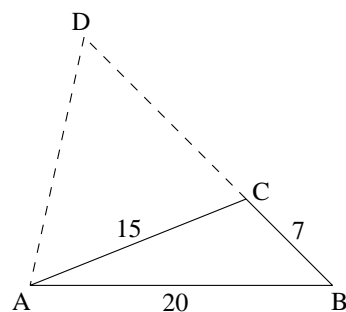
$$\text{Amy: } z = 1, u = 2 \quad \text{Bob: } y = 3, v = 2 \quad \text{Cathy: } z = 5, u = 3$$

$$\text{Dave: } x = 4, v = 2 \quad \text{Eva: } x = 4, y = 1$$

In their partial answers above, each of the contestants had found the correct value of exactly one unknown. What is the value of $x + z + v$?

- (a) 6 (b) 7 (c) 8 (d) 9 (e) 10

29. In $\triangle ABC$, $AB = 20$, $BC = 7$, and $CA = 15$. Side \overline{BC} is extended to point D so that $\triangle DAB$ is similar to $\triangle DCA$. What is DC ?



- (a) 9 (b) 10 (c) 11 (d) 12 (e) 13

30. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$, where a , b , c , and d are real numbers. Suppose the graph of $y = f(x)$ intersects the graph of $y = 2x - 1$ at $x = 1$, 2, and 3. What is the value of $f(0) + f(4)$?

- (a) 14 (b) 18 (c) 22 (d) 26 (e) 30