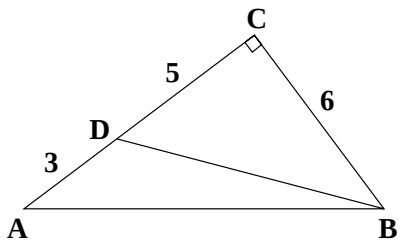
High School Math Contest
University of South Carolina
February 7, 2004

1. Given that $\angle ACB$ is a right angle, what is the area of $\triangle ABD$ in the figure shown?



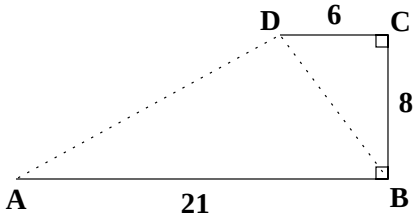
(a) 6 (b) 7 (c) 8 (d) 9 (e) 10

2. Which of the numbers below is a solution to the following equation?

$$\sqrt{3-x}+\sqrt{3+x}=x$$

(a) $\sqrt{6}$ (b) $2\sqrt{3}-1$ (c) $\dfrac{3\sqrt{3}}{2}$ (d) $\sqrt{6+\sqrt{2}}$ (e) $2\sqrt{2}$

3. What is the sum of the distances AD and BD in the figure shown?



(a) 27 (b) 28 (c) 29 (d) 30 (e) 31

4. What is the value of the product $(\log_2 3)\cdot(\log_3 5)\cdot(\log_5 8)$?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

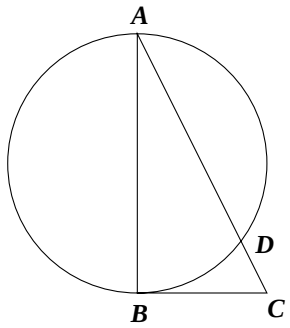
5. If $\sin x = 2\cos x$, then what is the value of $\sin x \cos x$?

(a) 1/3 (b) 2/3 (c) 1/4 (d) 1/5 (e) 2/5

6. Suppose that $f(x) = ax + b$ where a and b are real numbers. Given that $f(f(f(x))) = 8x + 21$, what is the value of $a + b$?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

7. Suppose $\overline{AB}$ is a diameter of the circle shown, $\overline{BC}$ is tangent to the circle, $\angle BAC = 30°$, and $CD = \sqrt{3}$. What is the distance from A to B ?



(a) $3\sqrt{3}$ (b) 6 (c) $4\sqrt{3}$ (d) 8 (e) $5\sqrt{3}$

8. A box contains 4 fair coins and 6 biased coins. Whenever a fair coin is flipped, it comes up heads with probability 0.5. Whenever a biased coin is flipped, it comes up heads with probability 0.8. A coin is randomly chosen from the box and then flipped. What is the probability that it will come up heads?

(a) 0.6 (b) 0.64 (c) 0.68 (d) 0.72 (e) 0.76

9. The expression $\left(\sqrt{2}\right)^{\log_2 9}$ simplifies to

(a) 3 (b) 4.5 (c) 6 (d) 7.5 (e) 9

10. If $a + b = 2$ and $a^2 + b^2 = 5$, then what is the value of $a^3 + b^3$?

(a) 11 (b) 12 (c) 13 (d) 14 (e) 15

11. How many different real roots does the following polynomial have?

$$x^{11} + x^{10} + x^9 + \cdots + x + 1$$

(a) 0 (b) 1 (c) 3 (d) 5 (e) 7

12. What is the ratio of the area of a square that circumscribes a circle to the area of a square that inscribes the same circle?

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) 2 (d) $\sqrt{6}$ (e) 3

13. Suppose that a, b, c, d, and e are numbers which satisfy the system of three equations below.

$$\begin{cases} 13a + 26b + 2c + 13d + 3e = 18 \\ 6a + 12b + c + 6d + e = 7 \\ 5a + 10b + c + 5d + e = 6 \end{cases}$$

What is the value of e ?

(a) -1 (b) 0 (c) 1 (d) 2 (e) 3

14. Suppose x is a complex number for which $x + \dfrac{1}{x} = 2\cos 12^\circ$. What is the value of $x^5 + \dfrac{1}{x^5}$?

(a) 4 (b) 3 (c) 2 (d) 1 (e) 0

15. What is the sum of all the different solutions to the following equation?

$$\frac{(x^2+1)(x^4+1)(x^6+1)}{x+1} + x - 1 = 0$$

(a) 4 (b) 3 (c) 2 (d) 1 (e) 0

16. For how many numbers $n \in \{1, 2, 3, \ldots, 2004\}$ is the quantity $4n^6 + n^3 + 5$ a multiple of 7 ?

(a) 0 (b) 285 (c) 286 (d) 287 (e) 2004

17. How many sequences a_1, a_2, a_3, a_4, a_5 satisfying $a_1 < a_2 < a_3 < a_4 < a_5$ can be formed if each a_i must be chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

(a) 118 (b) 120 (c) 122 (d) 124 (e) 126

18. How many different real numbered pairs (x, y) satisfy the system of two equations below?

$$\begin{cases} x + xy + y = -9 \\ x^2 + y^2 = 17 \end{cases}$$

(a) 6 (b) 4 (c) 3 (d) 2 (e) 0

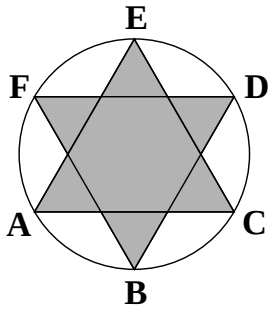
19. Suppose that a is a non-zero real number for which $\sin x + \sin y = a$ and $\cos x + \cos y = 2a$. What is the value of $\cos(x - y)$?

(a) $\frac{a^2 - 2}{2}$ (b) $\frac{3a^2 - 2}{2}$ (c) $\frac{5a^2 - 2}{2}$ (d) $\frac{7a^2 - 2}{2}$ (e) $\frac{9a^2 - 2}{2}$

20. Seven people are sitting in a theater watching a show. The row they are in contains seven seats. After intermission, they return to the same row but choose seats randomly. What is the probability that neither of the people sitting in the two aisle seats was previously sitting in an aisle seat?

(a) 3/7 (b) 10/21 (c) 11/21 (d) 4/7 (e) 13/21

21. The six points A, B, C, D, E, F are equally spaced along the circumference of a circle of radius 1. Line segments $\overline{AC}$, $\overline{CE}$, $\overline{AE}$, $\overline{BD}$, $\overline{DF}$, and $\overline{BF}$ are drawn, and the resulting star-shaped region is shaded in. What is the area of this star-shaped region?

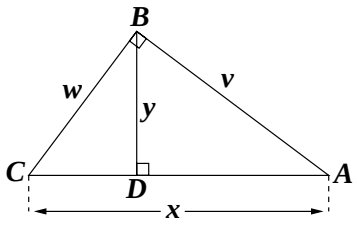


(a) $\frac{2\sqrt{6}}{3}$ (b) $\sqrt{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{3\sqrt{2}}{2}$ (e) $\frac{3\pi}{4}$

22. From a class of 12 students, 3 are chosen to form a math contest team. The team is required to include at least one boy and at least one girl. If exactly 160 different teams can be formed from the 12 students, then which of the following can be the difference between the number of boys and the number of girls in the class?

(a) 0 (b) 2 (c) 4 (d) 6 (e) 8

23. In the figure shown, $\angle ABC$ and $\angle BDA$ are both right angles. If $v+w = 35$ and $x+y = 37$, then what is the value of y ?



(a) 11 (b) 12 (c) 13 (d) 14 (e) 15

24. Suppose θ is an angle between 0° and 90° for which $\cos(\theta)\cos(2\theta) = 1/4$. What is the value of θ ?

(a) 32° (b) 34° (c) 36° (d) 38° (e) 40°

25. How many integers do the following finite arithmetic progressions have in common?

$$1, 8, 15, 22, \ldots, 2003 \qquad \text{and} \qquad 2, 13, 24, 35, \ldots, 2004$$

(a) 10 (b) 23 (c) 25 (d) 26 (e) 27

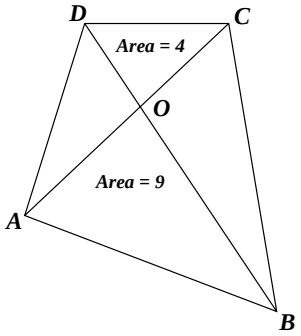
26. A standard, fair, 6-sided die is rolled 8 times. Given that the number 3 appears exactly three times, what is the probability that no two 3's appear on consecutive rolls?

(a) 5/14 (b) 3/7 (c) 1/2 (d) 4/7 (e) 9/14

27. A point (x, y) is called integral if both x and y are integers. How many integral points are inside of the parallelogram whose four sides are on the lines $x = 100$, $x = 300$, $y = \frac{1}{3}x + 0.1$, and $y = \frac{1}{3}x + 0.6$?

(a) 73 (b) 71 (c) 69 (d) 67 (e) 65

28. Let $ABCD$ be the quadrilateral shown, where O is the intersection of the line segments $\overline{AC}$ and $\overline{BD}$. The areas of $\triangle AOB$ and $\triangle COD$ are respectively 9 and 4. What is the minimum possible area of the quadrilateral $ABCD$?



(a) 22 (b) 23 (c) 24 (d) 25 (e) 26

29. Let a be the integer $\overbrace{333\cdots333}^{2003 \text{ digits}}$ and let b be the integer $\overbrace{666\cdots666}^{2003 \text{ digits}}$. What is the 2004th digit (counting from the right) that appears in the product ab ?

(a) 0 (b) 1 (c) 2 (d) 7 (e) 8

30. In how many ways can 2004 be written as a sum of two or more consecutive positive integers written in increasing order? For example, we consider $667 + 668 + 669$ as an acceptable sum but not $669 + 668 + 667$.

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5