SOLUTIONS TO USC'S 2003 HIGH SCHOOL MATH CONTEST

1. **(e)** Observing that $6^4 = 2^4 \times 3^4$, one can cancel to obtain $n = 3^4 = 81$.

2. **(c)** Three positive numbers are the lengths of the sides of a triangle if and only if each sum of two of the numbers is greater than the third. The choice 8 is the only one of the choices that satisfies this condition.

3. **(e)** Since 40% of 100 is $100 \cdot 0.4 = 40$ and $100 - 40 = 60$, the original sale price of the shirt is $\$60$. Since 20% of 60 is $60 \cdot 0.2 = 12$ and $60 - 12 = 48$, the shirt would cost the employee $\$48$.

4. **(d)** The three horizontal edges on the top of the figure have lengths summing to the length of the bottom edge, and the three vertical edges on the right of the figure have lengths summing to the length of the left edge. Thus, the perimeter is $2(12 + 16) = 56$.

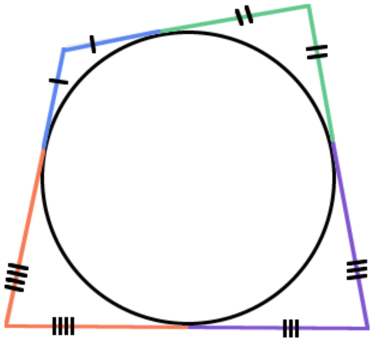
5. **(a)** The areas in the choices are (a) $\pi \cdot 3^2 > 27$, (b) $5^2 = 25$, (c) $3 \cdot 9 = 27$, (d) $6 \cdot 8/2 = 24$ (note that this is a right triangle with legs of lengths 6 and 8), and (e) less than $7^2/2 < 25$ (clearly, the height of the triangle is < 7).

6. **(b)** The answer follows from $81000000 < 87654321 < 100000000$.

7. **(d)** The solutions to the given equation are precisely the numbers satisfying one of the equations $x^2 + 4x - 2 = 5x^2 - 1$ and $x^2 + 4x - 2 = -\big(5x^2 - 1\big) = -5x^2 + 1$. The first of these can be rewritten as $4x^2 - 4x + 1 = (2x - 1)^2 = 0$ which has the one solution $x = 1/2$. The second can be rewritten as $6x^2 + 4x - 3 = 0$ which has two solutions (one can solve for them using the quadratic formula or one can note that the graph of $y = 6x^2 + 4x - 3$ is a parabola opening upward with y-intercept -3). Since $1/2$ is not a root of $6x^2 + 4x - 3$, there are 3 solutions (all real).

8. **(b)** This is the only choice that is less than one.

9. **(b)** Observe the segments of equal length as marked to the right. We deduce that the perimeter is simply twice the sum of the lengths of the two sides whose lengths are given. In other words, the perimeter of the quadrilateral is $2(10 + 16) = 52$. Note that typically a circle does not exist that is tangent to each of the sides of a quadrilateral. Thus, it is usually *not* the case that twice the sum of the lengths of opposite sides of a quadrilateral equals the perimeter of the quadrilateral.



10. **(e)** Let $A = 0.8$. The probability that all three choose the correct answer is $A \cdot A \cdot (1/2) = A^2/2$. The probability that Cathy and Bob choose the correct answer and Dave chooses the incorrect answer is $A \cdot A \cdot (1/2) = A^2/2$. The probability that Cathy and Dave choose the correct answer and Bob chooses the incorrect answer is $A \cdot (1/2) \cdot (1 - A) = A(1 - A)/2$. The probability that Bob and Dave choose the correct answer and Cathy chooses the incorrect answer is $A \cdot (1/2) \cdot (1 - A) = A(1 - A)/2$. It follows that the probability that the team response is correct is

$$\frac{1}{2}\big(A^2 + A^2 + A(1 - A) + A(1 - A)\big) = A.$$

Thus, the answer is $A = 80\%$. Observe that the above argument works for any value of A (that is, if 80% is replaced by A in the problem, the answer is A).

11. **(d)** Since $\lfloor\sqrt{x}\rfloor = 9$ and $\lfloor\sqrt{y}\rfloor = 12$, we deduce $x < 100$ and $y < 169$ so that $\lfloor x + y\rfloor < 269$. Since $x = 99.5$ and $y = 168.5$ satisfy $\lfloor\sqrt{x}\rfloor = 9$, $\lfloor\sqrt{y}\rfloor = 12$, and $\lfloor x + y\rfloor = 268$, the answer is 268.

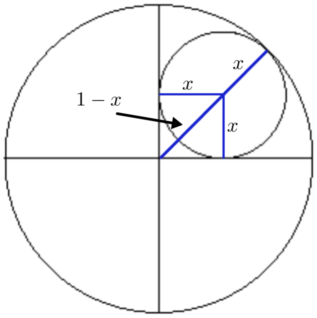
12. **(a)** Observe that the answer does not change if we take a and b to be positive. Suppose then that they are. Since $a^2 - b^2 = 2003$, we also have $a > b$. Since $2003 = a^2 - b^2 = (a - b)(a + b)$ and 2003 is prime, we deduce $a - b = 1$

and $a+b=2003$. Solving, we obtain $a=1002$ and $b=1001$. A direct computation gives $a^2+b^2=2006005$ (but this computation is not necessary since one easily sees that a^2+b^2 ends with a 5 which only choice (a) does).

13. **(c)** Let x denote the radius of the smaller circle. Then the figure has lengths as marked to the right. It follows that

$$x^2+x^2=(1-x)^2 \implies x^2+2x-1=0$$
$$\implies x=\pm\sqrt{2}-1.$$

Since $x>0$, we obtain $x=\sqrt{2}-1$.



14. **(b)** The sum of the roots of a polynomial of degree n is equal to minus the coefficient of x^{n-1} divided by the coefficient of x^n. The coefficient of x^{10} in $f(x)$ is 1 so that the sum of the roots of $f(x)$ is minus the coefficient of x^9 in $f(x)$. This coefficient is the sum of the coefficient of x^9 in $(x-9)^9$, which is 1, and the coefficient of x^9 in $(x-10)^{10}$, which is -100. Thus, the answer is $-\big(1-100\big)=99$.

15. **(a)** The first equation implies $y=1/y$ so that $y^2=1$. If $y=1$, then the second equation in the problem implies $x=x-1$, which is impossible. Hence, $y=-1$. Now, the second equation gives $-x=x+1$ so that $x=-1/2$. Thus, $x+y=-3/2$.

16. **(d)** Let $t=1+\log_3 x$. Converting the given equation so that it involves logarithms to the base 3, we deduce that

$$\frac{\log_3 3}{\log_3 3+\log_3 x}+\frac{\log_3 3+\log_3 x}{\log_3 27}=-\frac{4}{3} \iff \frac{1}{t}+\frac{t}{3}=-\frac{4}{3} \iff t^2+4t+3=0 \iff (t+1)(t+3)=0.$$

We deduce that $2+\log_3 x=t+1=0$ or $4+\log_3 x=t+3=0$ so that $\log_3 x$ is either -2 or -4 and x is either $1/9$ or $1/81$. Thus, the answer is $1/9+1/81=10/81$.

17. **(c)** Let $x=3\theta$. The problem then is to determine for how many values of $x\in[0,3\pi]$ does $\cos(5x)=\cos x$. The equality can occur if and only if $5x=2\pi k\pm x$ for some integer k. The equation $5x=2\pi k+x$ is equivalent to $x=\pi k/2$, and the equation $5x=2\pi k-x$ is equivalent to $x=\pi k/3$. The multiples of π are the values of x which are in both of these forms and there are 4 of them in $[0,3\pi]$. There are an additional 3 values of x in $[0,3\pi]$ corresponding to the equation $x=\pi k/2$ and an additional 6 values of x in $[0,3\pi]$ corresponding to the equation $x=\pi k/3$. Thus, the answer is $4+3+6=13$.

18. **(a)** Let O be the center of the circle, and let r be its radius. We use that the area of a triangle is one-half the product of the lengths of any two sides of the triangle times the sine of the (smallest) angle formed by the two sides. Since the sum of the areas of $\triangle AOB$, $\triangle BOC$, and $\triangle COA$ is the area of $\triangle ABC$, which is 8, we deduce

$$\frac{1}{2}r^2\sin\alpha+\frac{1}{2}r^2\sin\beta+\frac{1}{2}r^2\sin\gamma=8 \implies \sin\alpha+\sin\beta+\sin\gamma=\frac{16}{r^2}.$$

Since the given circle has area 20, we deduce that $\pi r^2=20$. The answer $16\pi/20=4\pi/5$ follows.

19. **(e)** Let $x=a+b+c$. Summing the three equations, we deduce that $x+x^2=72$, so $(x-8)(x+9)=0$. Since a, b, and c are positive, we deduce that $x>0$ so that $a+b+c=8$. Note that $a=4$, $b=1$, and $c=3$ satisfy the three equations.

20. **(d)** Since $1-(1/n^2)=(n-1)(n+1)/n^2$, the product is equivalent to

$$\frac{1\cdot 3}{2^2}\times\frac{2\cdot 4}{3^2}\times\frac{3\cdot 5}{4^2}\times\frac{4\cdot 6}{5^2}\times\frac{5\cdot 7}{6^2}\times\cdots\times\frac{1999\cdot 2001}{2000^2}\times\frac{2000\cdot 2002}{2001^2}\times\frac{2001\cdot 2003}{2002^2}\times\frac{2002\cdot 2004}{2003^2}.$$

Cancelling like factors in the numerators and denominators, we see that the product is $2004/\big(2\cdot 2003\big)=1002/2003$.

21. **(e)** Let $u=f(2)$ and $v=f(-1/2)$. Then setting $x=1/2$ and $x=-2$, we obtain $u+2v=1$ and $v-(1/2)u=-4$. Subtracting twice the second of these from the first and solving for u, we deduce $f(2)=u=9/2=4.5$. Note that the function $f(x)=(x^3+1)/x$ satisfies the conditions of the problem.

22. **(d)** Recall $\tan(x+y) = \big(\tan x + \tan y\big)/\big(1 - \tan x \tan y\big)$. With $x = \arctan(1/2)$ and $y = \arctan(1/3)$, we deduce $\tan(x+y) = 1$ so that $x + y = \pi/4$ (as both x and y are clearly in $[0, \pi/2]$). So the problem is to determine the value of $\sin^2(\pi/4)$, which is easily seen to be $1/2$.

23. **(c)** Let $N = (100x + 10y + z)^2 = (x + y + z)^5$. Then N is both a square and a fifth power implying that N must be a tenth power. Since N is the square of a 3 digit number and

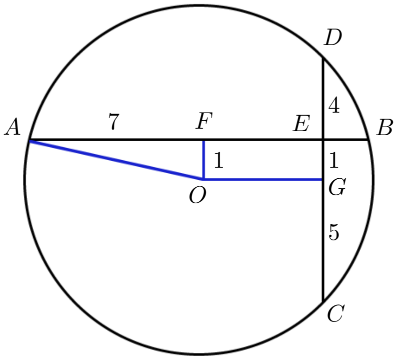
$$2^{10} = (2^5)^2 = 32^2, \quad 3^{10} = (3^5)^2 = 243^2, \quad \text{and} \quad 4^{10} = (4^5)^2 = 1024^2,$$

we deduce $N = 243^2$ (other choices are either too small or too large). Thus, $x = 2$, $y = 4$, and $z = 3$ are the only possible values of these variables. One checks that in fact the equation holds for these values of the variables. Hence, $x^2 + y^2 + z^2 = 4 + 16 + 9 = 29$.

24. **(c)** Since N divided by 5 gives a remainder of 2, we have $N = 5q + 2$ for some integer q (the quotient when dividing N by 5). Then $2N + 1 = 10q + 5$ so that 5 divides $2N + 1$. Similarly, the given information implies that $2N + 1$ is also divisible by both 7 and 9. We deduce that $2N + 1$ is divisible by $5 \cdot 7 \cdot 9 = 315$. Thus, $2N + 1 \geq 315$, which implies $N \geq 157$. Since 157 satisfies the conditions in the problem, $N = 157$ and the sum of its digits is 13.

25. **(d)** If $n^2 + 4$ and $n + 3$ are both divisible by d, then so is $n^2 + 4 - (n+3)(n-3) = 13$. Thus, if $n^2 + 4$ and $n + 3$ have a common factor > 1, then it is 13. Observe that $n + 3$ is divisible by 13 precisely when $n = 13k + 10$ for some integer k. Also, $(13k + 10)^2 + 4 = 169k^2 + 260k + 104 = 13\big(13k^2 + 20k + 8\big)$ is divisible by 13. Since $13k + 10$ is between 1 and 100 if and only if $k \in \{0, 1, \ldots, 6\}$, the answer is 7.

26. **(a)** Given the two chords $\overline{AB}$ and $\overline{CD}$ intersect at E, one has $AE \cdot BE = CE \cdot DE$. (This result from Geometry can be shown by observing that the triangles $\triangle ACE$ and $\triangle DBE$ are similar so that $AE/CE = DE/BE$, implying $AE \cdot BE = CE \cdot DE$.) The given information implies $BE = 6 \cdot 4/12 = 2$. Let O denote the center of the circle. Let F be the midpoint of segment $\overline{AB}$ so that $\overleftrightarrow{OF}$ is perpendicular to $\overleftrightarrow{AB}$, and let G be the midpoint on segment $\overline{DC}$ so that $\overleftrightarrow{OG}$ is perpendicular to $\overleftrightarrow{DC}$. Then $AF = 7$. Also, $DG = 5$ so that $OF = GE = 1$. It follows that $OA^2 = 7^2 + 1 = 50$ so that the area of the circle is 50π.



27. **(b)** For $n \leq 4$, one checks that $3^n + 81$ is not a square. Suppose now that $n = k + 4$ where k is a positive integer. Then $3^n + 81 = 81(3^k + 1)$. Since 81 is a square, we deduce that $3^n + 81$ is a square if and only if there is a positive integer x such that $3^k + 1 = x^2$. On the other hand, $3^k + 1 = x^2$ if and only if $(x-1)(x+1) = x^2 - 1 = 3^k$, which holds for some k precisely when both $x - 1$ and $x + 1$ are powers of 3. The only two powers of 3 differing by 2 are 1 and 3 so necessarily $x = 2$ and $k = 1$. Thus, $n = k + 4 = 5$ is the only positive integer for which $3^n + 81$ is a square.

28. **(c)** We determine first the common sum, say S, of the numbers in each circle. We make use of the fact that $a+b+c+\cdots+i = 1 + 2 + 3 + \cdots + 9 = 45$. Taking the nine sums, one sum for each circle, and adding them together, we deduce

$$9S = 1 + 2 + 3 + \cdots + 9 + 2\big(a + b + c + \cdots + i\big) = 3 \times 45 \implies S = 15.$$

Next, we take six sums, adding the numbers in the six circles with center numbers 1, 2, 4, 5, 7, and 8. This gives

$$90 = 6S = 1 + 2 + 4 + 5 + 7 + 8 + (a + b + c + \cdots + i) + a + d + g = 27 + 45 + a + d + g = 72 + a + d + g.$$

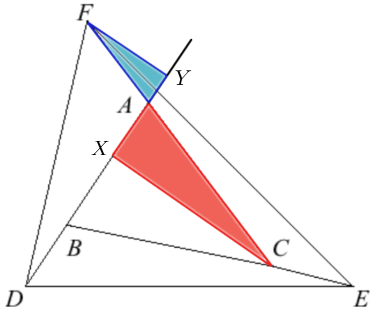
We deduce that $a + d + g = 90 - 72 = 18$. Alternatively, one can guess that putting the largest number 9 between the circles with center numbers 1 and 2 (or the smallest number 1 between the circles with center numbers 8 and 9) leads to a correct substitution of the letters $a, b, c, \ldots$ with the numbers $1, 2, 3, \ldots$. Using $S = 15$, we can obtain quickly that $a = 9$, $b = 4$, $c = 8$, $d = 3$, $e = 7$, $f = 2$, $g = 6$, $h = 1$, and $i = 5$ is such a substitution. This is in fact the only correct substitution for this problem.

29. **(b)** Let q and r be the quotient and remainder when dividing n by 3, so $n = 3q + r$ and $r \in \{0, 1, 2\}$. Since $5^3 - 1 = (5-1)(5^2 + 5 + 1) = 4 \cdot 31$, we deduce

$$5^n - 5^r = 5^{3q+r} - 5^r = 5^r\left(5^{3q} - 1\right) = 5^r\left(5^3 - 1\right)\left(5^{3q-3} + 5^{3q-6} + \cdots + 5^3 + 1\right) = 31k,$$

for some integer k. In order for $5^n + n$ to be divisible by 31, it is then both necessary and sufficient that $5^r + n$ (the difference $(5^n + n) - (5^n - 5^r)$) be divisible by 31. We consider each $r \in \{0, 1, 2\}$. If $r = 0$, then $n = 3q$ and we want $5^r + n = 1 + n$ to be divisible by 31. Clearly, $n = 30$ is the smallest such n. If $r = 1$, then $n = 3q + 1$ and we want $5^r + n = 5 + n$ to be divisible by 31. One checks that 88 is the smallest such n. If $r = 2$, then $n = 3q + 2$ and we want $5^r + n = 25 + n$ divisible by 31. The smallest such n here is $n = 68$. Thus, the answer is $n = 30$.

30. **(b)** Let X be the point on $\overline{AB}$ such that CX is an altitude of $\triangle ABC$. Let Y be the point on $\overleftrightarrow{AD}$ such that FY is an altitude of $\triangle ADF$. Then $\triangle AXC$ is similar to $\triangle AYF$ so that $FY/CX = AF/AC = 1/2$. Letting $\mathcal{A}(x)$ denote the area of a triangle x, we deduce



$$\begin{aligned}\mathcal{A}(\triangle ADF) &= \frac{1}{2} \cdot AD \cdot FY = \frac{1}{2} \cdot \frac{3}{2}AB \cdot \frac{1}{2}CX \\ &= \frac{3}{4}\left(\frac{1}{2} \cdot AB \cdot CX\right) = \frac{3}{4}\mathcal{A}(\triangle ABC).\end{aligned}$$

Similarly,

$$\mathcal{A}(\triangle BED) = (3/4)\mathcal{A}(\triangle ABC) \quad \text{and} \quad \mathcal{A}(\triangle CFE) = (3/4)\mathcal{A}(\triangle ABC).$$

Thus,

$$\begin{aligned}\mathcal{A}(\triangle DEF) &= \mathcal{A}(\triangle ABC) + \mathcal{A}(\triangle ADF) + \mathcal{A}(\triangle BED) + \mathcal{A}(\triangle CFE) \\ &= \left(1 + \frac{3}{4} + \frac{3}{4} + \frac{3}{4}\right)\mathcal{A}(\triangle ABC) = \frac{13}{4}\mathcal{A}(\triangle ABC).\end{aligned}$$

The requested ratio is therefore $13/4$.