High School Math Contest
University of South Carolina
January 20, 2001

1. What is the value of $\dfrac{5}{6^{-2} \cdot 8^{1/3}}$?

   (a) 60 (b) 70 (c) 80 (d) 90 (e) 100

2. A square has perimeter $p > 0$ and area A. If $A = 2p$, then what is the value of p ?

   (a) 24 (b) 32 (c) 36 (d) 48 (e) 54

3. If $\theta = 11^\circ$, then what is the value of $(\sin\theta + \cos\theta)^2 - \sin(2\theta)$?

   (a) $\dfrac{1}{2}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) $\dfrac{1+\sqrt{5}}{2}$ (d) 1 (e) 0

4. If the length of each side of a triangle is increased by 20%, then the area of the triangle is increased by

   (a) 40% (b) 44% (c) 48% (d) 52% (e) 60%

5. Given that the vertex of the parabola $y = x^2 + 8x + k$ is on the x-axis, what is the value of k ?

   (a) 0 (b) 4 (c) 8 (d) 16 (e) 24

6. The radius of a circular pond is 30 feet, the radius of a circular lake is 700 feet, and the center of the lake is 600 feet east and 800 feet north of the center of the pond. If a baby duck walks from the edge of the pond to the edge of the lake, then what is the shortest distance that he must walk?

   (a) 260 feet (b) 270 feet (c) 280 feet (d) 290 feet (e) 300 feet

7. If $9^{-x} = 7$, then what is the value of 27^{2x+1} ?

(a) $\dfrac{27}{7\sqrt{7}}$ (b) $189\sqrt{7}$ (c) $\dfrac{343}{27}$ (d) $\dfrac{7\sqrt{7}}{27}$ (e) $\dfrac{27}{343}$

8. There are 29 people in a room. Of these, 11 speak French, 24 speak English and 3 speak *neither* French nor English. How many people in the room speak *both* French and English?

(a) 3 (b) 4 (c) 6 (d) 8 (e) 9

9. Let x and y be real numbers such that $(x^2 - y^2)(x^2 - 2xy + y^2) = 3$ and $x - y = 1$. What is the value of xy ?

(a) 2 (b) $1 + \sqrt{2}$ (c) $1 - \sqrt{2}$ (d) 1 (e) 0

10. How many occurrences of the digit 5 are there in the list of numbers $1, 2, 3, \ldots, 1000$?

(a) 200 (b) 300 (c) 333 (d) 385 (e) 500

11. Which one of the following numbers is nearest in value to the quantity $\sqrt{101} - 10$?

(a) $\dfrac{1}{16}$ (b) $\dfrac{1}{18}$ (c) $\dfrac{1}{20}$ (d) $\dfrac{1}{22}$ (e) $\dfrac{1}{24}$

12. Solve for x in the equation

$$\log_4 \sqrt{x^{4/3}} + 3 \log_x (16x) = 7.$$

(a) 16 (b) 27 (c) 64 (d) 81 (e) 343

13. How many of the following statements are true?

- There exists an even prime number.
- The number $2^{65} + 1$ is prime.
- There exist distinct integers m and n such that $m^2 = n^3$.
- Some quadratic equations have no real solutions.
- The cubic equation $x^3 + x^2 + 1 = 0$ has a real solution.

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

14. Wilson's Theorem states that if n is a prime number, then n divides $(n-1)! + 1$. Which of the following is a divisor of $12! \cdot 6! + 12! + 6! + 1$?

(a) 21 (b) 77 (c) 91 (d) 115 (e) 143

15. Given the polynomial identity

$$x^6 + 1 = (x^2 + 1)(x^2 + ax + 1)(x^2 + bx + 1),$$

what is the value of ab ?

(a) -3 (b) -1 (c) 0 (d) 1 (e) 5

16. For how many integers m, with $10 \leq m \leq 100$, is m^2+m-90 divisible by 17 ?

(a) 7 (b) 8 (c) 9 (d) 10 (e) 11

17. Three standard, fair, 6-sided dice are tossed simultaneously. What is the probability that the numbers shown on some two of the dice add up to give the number shown on the remaining die?

(a) 5/36 (b) 1/6 (c) 7/36 (d) 2/9 (e) 5/24

18. What is the number of *distinct* real solutions to the equation

$$x^4 + 6x^2 + 9 = 36x^2 - 72x + 36 \ ?$$

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

19. A little old lady is driving on a straight road at a constant speed. She is XYZ miles from her home in Pasadena at 2 o'clock, where X, Y, and Z are digits with $X \geq 1$ and $Y = 0$. At 2:18, she is ZX miles from home, and at 3:00 she is XZ miles from home. At what time does she arrive home?

(a) 3:10 (b) 3:12 (c) 3:24 (d) 3:30 (e) 3:48

20. Beginning at 5:00 P.M., how many hours must elapse before the hour-hand and minute-hand of a clock are perpendicular to each other?

(a) 1/5 (b) 2/11 (c) 5/22 (d) 4/23 (e) 7/30

21. A drawer contains 64 socks. Each sock is one of 8 colors, and there are 8 socks of each color. If the socks in the drawer are thoroughly mixed and you randomly choose two of the socks, then what is the probability that these two socks will have the same color?

(a) 1/7 (b) 1/8 (c) 1/9 (d) 7/64 (e) 9/64

22. If the number 2001 is written in the form

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (n-2) - (n-1) + n,$$

then what is the sum of the digits of n ?

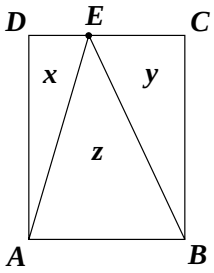
(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

23. In the sum below, the letter $F = 0$, and the other letters represent the digits 1, 2, 3, 4, 5, or 6, with each digit used exactly once. The 2-digit integer AB is a prime number. What is the value of $A + B$?

$$\begin{array}{r} AB \\ +\ CD \\ \hline EFG \end{array}$$

(a) 3 (b) 4 (c) 5 (d) 7 (e) 9

24. Suppose that $ABCD$ is a rectangle, and that E is a point on $\overline{CD}$. Let x be the area of $\triangle AED$, y be the area of $\triangle BCE$, and z be the area of $\triangle ABE$, and suppose that $y^2 = xz$. What is the value of $\dfrac{DE}{EC}$?



(a) $\dfrac{3}{5}$ (b) $\dfrac{\sqrt{5}-1}{2}$ (c) $\dfrac{2}{3}$ (d) $\dfrac{\sqrt{5}}{3}$ (e) $\dfrac{\sqrt{3}}{2}$

25. Each of the following five statements is either true or false.

(1) Statements (3) and (4) are both true.
(2) Statements (4) and (5) are not both false.
(3) Statement (1) is true.
(4) Statement (3) is false.
(5) Statements (1) and (3) are both false.

How many of statements (1) — (5) are true?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

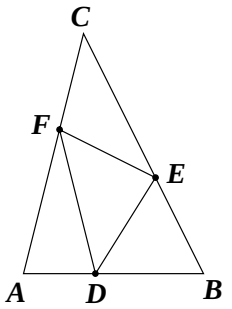
26. What is the value of $\sin 10^\circ \sin 50^\circ \sin 70^\circ$?

(a) $\dfrac{1}{9}$ (b) $\dfrac{1}{8}$ (c) $\dfrac{\sqrt{3}}{12}$ (d) $\dfrac{\sqrt{2}}{8}$ (e) $\dfrac{1}{6}$

27. Two intersecting circles each have radius 6, and the distance between the centers of the circles is $6\sqrt{3}$. Find the area of the region that lies inside both circles.

(a) $2\pi - \sqrt{3}$ (b) $6\pi - 4\sqrt{3}$ (c) $6\pi - 12\sqrt{3}$
(d) $12\pi - 18\sqrt{3}$ (e) $12\pi - 24\sqrt{3}$

28. On $\triangle ABC$, point D lies on $\overline{AB}$, point E lies on $\overline{BC}$, and point F lies on $\overline{CA}$. If $\dfrac{AD}{DB} = \dfrac{BE}{EC} = \dfrac{CF}{FA} = \dfrac{2}{3}$, and the area of $\triangle ABC$ equals 1, then what is the area of $\triangle DEF$?

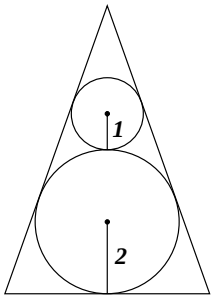


(a) $\dfrac{3}{25}$ (b) $\dfrac{4}{9}$ (c) $\dfrac{5}{9}$ (d) $\dfrac{8}{25}$ (e) $\dfrac{7}{25}$

29. What is the smallest positive integer n such that if S is any set containing n or more integers, then there must be three integers in S whose sum is divisible by 3 ?

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

30. In an isosceles triangle, the inscribed circle has radius 2. Another circle of radius 1 is tangent to the inscribed circle and the two equal sides. What is the area of the triangle?



(a) $16\sqrt{2}$ (b) 20 (c) $13\sqrt{2}$ (d) $11\sqrt{3}$ (e) 21.5