SOLUTIONS TO USC'S 2000 HIGH SCHOOL MATH CONTEST

1. **(a)** Since $2^3 \times 3^2 x^2 = 72x^2 = 9800 = 2^3 \times 5^2 \times 7^2$, we obtain $x^2 = 5^2 \times 7^2/3^2$. Since $x > 0$, we deduce that $x = 5 \times 7/3 = 35/3$.

2. **(b)** Since $\log_2(\log_2(\log_2 16)) = \log_2(\log_2 4) = \log_2 2 = 1$, the answer is 1.

3. **(e)** Using that $2^{600} = 64^{100}$, $3^{500} = 243^{100}$, $4^{400} = 256^{100}$, $5^{300} = 125^{100}$, and $6^{200} = 36^{100}$, we see that the answer is 6^{200}.

4. **(b)** Let $x = 35811231$ and observe that $0 < x < 10^8$. Hence,

$$\left(\frac{10^{14} + x}{10^{14}}\right)^2 = \frac{10^{28} + (2x)10^{14} + x^2}{10^{28}} = \frac{100000071622462 \times 10^{14} + x^2}{10^{28}}.$$

Since $0 < x^2 < 10^{16}$, we see that the expression x^2 does not effect the leading 12 digits of the integer in the numerator. Thus, $1.00000035811231^2 = 1.000000716224\ldots$. Hence, $x = 7$, $y = 1$, and $z = 6$ so that $x + y + z = 14$.

5. **(e)** Given

$$(x^3 - x^2 - 5x - 2)(x^4 + x^3 + kx^2 - 5x + 2) = x^7 - 4x^5 - 14x^4 - 5x^3 + 19x^2 - 4,$$

the coefficient of x^2 on the left is $-2k + 25 - 2 = -2k + 23$ and the coefficient of x^2 on the right is 19. These must be equal so that $-2k + 23 = 19$. Hence, $k = 2$.

6. **(d)** The 100 horizontal lines divide the plane into 101 regions. If each vertical line is considered in turn, each divides the plane into 101 additional regions. The total number of regions obtained is $101^2 = 10201$.

7. **(c)** The answer is $2^{15} = 32768$ since it is $\left(2^3\right)^5 = 8^5$ as well as $\left(2^5\right)^3 = 32^3$. To see that N cannot be smaller, observe that the conditions in the problem imply $N = a^5$ for some positive integer a and that for $1 \le a \le 7$ the number a^5 is not a cube of a number different from a.

8. **(d)** Recall that $2\sin\theta\cos\theta = \sin(2\theta)$. Hence,

$$(\sin 15^\circ)^2(\cos 15^\circ)^2 = (\sin 15^\circ \cos 15^\circ)^2 = \left(\frac{\sin 30^\circ}{2}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

9. **(a)** Let $x = BC$. Since ΔDBC is a 45-90-45 degree triangle, $DB = x$. Since ΔABC is a 30-90-60 degree triangle and $\tan 30^\circ = 1/\sqrt{3}$, we deduce that

$$\frac{1}{\sqrt{3}} = \tan 30^\circ = \frac{BC}{AB} = \frac{BC}{AD + DB} = \frac{x}{2 + x}.$$

Therefore, $2 + x = x\sqrt{3}$, from which we deduce that $BC = x = 2/(\sqrt{3} - 1) = \sqrt{3} + 1$.

10. **(b)** Since $9^3 = 729$ and $10^3 = 1000$, we see that $\sqrt[3]{829} = 9 + r$ where $0 < r < 1$. Since $10^2 = 100$ and $10^3 = 1000$, we see that $\log_{10} 829 = 2 + s$ where $0 < s < 1$. Thus,

$$\frac{\sqrt[3]{829} + \log_{10} 829}{2} = \frac{(9 + r) + (2 + s)}{2} = \frac{11 + (r + s)}{2} = 5.5 + t,$$

where $0 < t = (r + s)/2 < 1$. In other words, $(\sqrt[3]{829} + \log_{10} 829)/2$ is strictly between 5.5 and 6.5. The nearest integer is therefore 6.

11. **(d)** The graph of $4x^2-9y^2=36$ is an hyperbola with vertices at $(-3,0)$ and $(3,0)$, and the graph of $x^2-2x+y^2=15$ is a circle of radius 4 centered at $(1,0)$ and therefore crossing the x-axis at $(-3,0)$ and $(5,0)$. It follows that these graphs intersect at three points.

Alternatively, if (a,b) is an intersection point, then $a^2-2a+b^2=15$ so that $9a^2-18a+9b^2=135$. Since also $4a^2-9b^2=36$, we deduce (by adding these equations) that $13a^2-18a-171=0$. In other words, $(a+3)(13a-57)=0$. Hence, $a=-3$ or $a=57/13$. Using the equation $4a^2-9b^2=36$, we see that in the first case $b=0$ and in the second case $b=\pm 16\sqrt{3}/13$. Therefore, there are three intersection points, $(-3,0)$ and $(57/13,\pm 16\sqrt{3}/13)$.

12. **(d)** Using $a_1=1$, a direct computation gives $a_2=1$, $a_3=0$, $a_4=0$, $a_5=2$, and $a_6=1$. Here, the process begins to repeat as 6 is even and $a_6=1$ (just as 2 is even and $a_2=1$). One deduces that for every positive integer k, $a_{4k}=0$, $a_{4k+1}=2$, $a_{4k+2}=1$, and $a_{4k+3}=0$. Thus, $a_j=2$ precisely when $j=5,9,13,17,\ldots$. For $1\le j\le 100$, there are 24 such j (the last one being $4\times 24+1=97$).

13. **(c)** The number of real roots of $x^4+|x|=10$ is the same as the number of intersection points of the graphs of $y=|x|$ and $y=10-x^4$. One easily deduces that these graphs intersect in exactly two points. Thus, there are a total of two real roots.

Alternately, if $x\ge 0$ and $x^4+|x|=10$, then $x^4+x=10$ so that $x^4+x-10=0$. This has one positive real solution by Descartes' Rule of Signs (and clearly 0 is not a solution). If $x<0$ and $x^4+|x|=10$, then $x^4-x=10$ so that $x^4-x-10=0$. Since $(-x)^4-(-x)-10=x^4+x-10$, the equation $x^4-x-10=0$ has exactly one negative real root by Descartes' Rule of Signs (alternatively, one can use that the negative real roots of $x^4-x-10=0$ are precisely minus the positive real roots of $x^4+x-10=0$). Thus, again, there are a total of two real roots.

14. **(e)** Factoring the left-hand side of the equation $x^2-xy+x-y=0$, we obtain $(x+1)(x-y)=0$. The latter is satisfied by all points (x,y) satisfying at least one of $x+1=0$ and $x-y=0$. In other words, the graph of $x^2-xy+x-y=0$ is composed of the two lines determined by the equations $x+1=0$ and $x-y=0$ (intersecting at $(-1,-1)$).

15. **(b)** The two shaded triangles are similar, and the ratio of the length of each side of the larger triangle to the length of the corresponding side of the smaller triangle is 3. It follows that the height of the larger triangle to base $\overline{AD}$ divided by the height of the smaller triangle to base $\overline{BC}$ is 3. Since the sum of the two heights is 5, one deduces that the height of the larger triangle is $15/4$ and the height of the smaller triangle is $5/4$. It follows that the sum of the two areas is

$$\frac{1}{2}\times 3\times\frac{15}{4}+\frac{1}{2}\times 1\times\frac{5}{4}=\frac{50}{8}=\frac{25}{4}=6.25.$$

16. **(c)** Let

$$f(x)=(1+x)^{20}+x(1+x)^{19}+x^2(1+x)^{18}+\cdots+x^{18}(1+x)^2.$$

The polynomial $f(x)$ does not change if we multiply it by $1=(1+x)-x$. Thus,

$$\begin{aligned} f(x)&=\big((1+x)-x\big)\big((1+x)^{20}+x(1+x)^{19}+x^2(1+x)^{18}+\cdots+x^{18}(1+x)^2\big)\\ &=(1+x)^{21}-x^{19}(1+x)^2. \end{aligned}$$

The coefficient of x^{18} when the expression $x^{19}(1+x)^2$ is expanded is clearly 0 (there is no x^{18} term), so the coefficient of x^{18} in $f(x)$ is simply the coefficient of x^{18} in $(1+x)^{21}$. By the binomial theorem, this is

$$\binom{21}{18}=\frac{21\times 20\times 19}{6}=7\times 10\times 19=1330.$$

Alternatively, one can use a more direct approach. The coefficient of x^{18} when the expression $x^{20-k}(1+x)^k$ is expanded is $\binom{k}{k-2}=\dfrac{k(k-1)}{2}$ where $2\le k\le 20$. The answer can be obtained by summing the numbers $k(k-1)/2$ from $k=2$ to $k=20$. This sum can be done more efficiently by using known formulas for $\sum_{k=1}^{n}k$

and $\sum_{k=1}^{n} k^2$. (By the way, if you do not know these formulas, you might consider taking $f(x) = (1+x)^n + x(1+x)^{n-1} + x^2(1+x)^{n-2} + \cdots + x^{n-1}(1+x) + x^n$ and using the first argument above to derive them by looking at the coefficients of x^{n-1} and x^{n-2}.)

17. **(e)** Say the cowboys names are Bob, Dave, Kevin, and Michael. Suppose exactly two of them were shot. There are $\binom{4}{2} = 6$ ways of choosing two of them, each equally likely. Say the two shot were Dave and Kevin. The probability that any one cowboy shot some other given cowboy is $1/3$. Since Dave and Kevin were the only two shot, Kevin must have shot Dave and Dave must have shot Kevin and each of these occurred with probability $1/3$. Michael could have shot Dave or Kevin, and he shot one of them with probability $2/3$. Similarly, Bob shot Dave or Kevin with probability $2/3$. Therefore, the probability that Dave and Kevin both were shot and no one else was is

$$\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{81}.$$

Taking into account that there are 6 ways of choosing two of the four cowboys, we deduce that the probability of exactly two cowboys being shot is $6 \times 4/81 = 8/27$.

18. **(d)** Observe that

$$\begin{aligned} 2N &= 3N - N \\ &= (k_1 \times 10^6 + k_2 \times 10^5 + \cdots + k_7) - (k_1 + k_2 \times 10 + \cdots + k_7 \times 10^6) \\ &= (10^6 - 1)(k_1 - k_7) + (10^5 - 10)(k_2 - k_6) + (10^4 - 10^2)(k_3 - k_5). \end{aligned}$$

Since $10^k - 10^\ell$ is divisible by 9 for any non-negative integers k and ℓ, we see that the last expression above is divisible by 9. It follows that $2N$ and, hence, N must be divisible by 9. One checks that the only choice of answers divisible by 9 is 71053290. Note that this does not prove that there are k_j as in the problem for $N = 71053290$ (such a proof is not required). For $N = 71053290$, one can take $k_1 = 210$, $k_2 = 28$, $k_3 = 28$, $k_4 = 70$, $k_5 = 98$, $k_6 = 0$, and $k_7 = 70$.

19. **(c)** The inequalities

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{4n+1}} < \sqrt{n} - \sqrt{n-1}$$

imply

$$\begin{aligned} \sum_{n=1}^{24} \frac{1}{\sqrt{4n+1}} &< \sum_{n=1}^{24} \left(\sqrt{n} - \sqrt{n-1}\right) \\ &= \left(\sqrt{1} - \sqrt{0}\right) + \left(\sqrt{2} - \sqrt{1}\right) + \left(\sqrt{3} - \sqrt{2}\right) + \left(\sqrt{4} - \sqrt{3}\right) + \cdots + \left(\sqrt{24} - \sqrt{23}\right) \\ &= \sqrt{24} < 5 \end{aligned}$$

and

$$\begin{aligned} \sum_{n=1}^{24} \frac{1}{\sqrt{4n+1}} &> \sum_{n=1}^{24} \left(\sqrt{n+1} - \sqrt{n}\right) \\ &= \left(\sqrt{2} - \sqrt{1}\right) + \left(\sqrt{3} - \sqrt{2}\right) + \left(\sqrt{4} - \sqrt{3}\right) + \left(\sqrt{5} - \sqrt{4}\right) + \cdots + \left(\sqrt{25} - \sqrt{24}\right) \\ &= \sqrt{25} - \sqrt{1} = 5 - 1 = 4. \end{aligned}$$

Hence, the answer is 4.

20. **(b)** Setting $P = (x, y)$, we see that

$$\begin{aligned} d &= (x+5)^2 + (y+1)^2 + (x+1)^2 + y^2 + (x-1)^2 + (y-2)^2 + (x-1)^2 + (y-3)^2 \\ &= 4x^2 + 8x + 4y^2 - 8y + 42 \\ &= 4(x+1)^2 + 4(y-1)^2 + 34. \end{aligned}$$

Since $(x+1)^2 \geq 0$ for all x with equality precisely when $x = -1$ and since $(y-1)^2 \geq 0$ for all y with equality precisely when $y = 1$, the minimum value of d is 34 (obtained when $x = -1$ and $y = 1$).

21. **(b)** By the definition of the logarithm, the solutions of $\log_x(5x-2) = 3$ for $x > 2/5$ correspond to x for which $5x - 2 = x^3$. Thus, we consider roots of the polynomial $f(x) = x^3 - 5x + 2$. Since $f(x)$ is a cubic, $f(x)$ has at most 3 roots. One easily checks that $f(-100) < 0$, $f(2/5) > 0$, $f(1) < 0$, and $f(100) > 0$. This implies that the graph of $y = f(x)$ crosses the x-axis somewhere in each of the intervals $(-100, 2/5)$, $(2/5, 1)$, and $(1, 100)$. Hence, each of these three intervals contains a root of $f(x)$. It follows that $f(x)$ has exactly two roots $> 2/5$.

22. **(d)** The conditions in the problem imply

$$x^3 - 64x - 14 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + \cdots - abc.$$

Hence, $a+b+c = 0$. Also, since a, b, and c are roots of $x^3 - 64x - 14$, we obtain

$$a^3 = 64a + 14, \qquad b^3 = 64b + 14, \qquad \text{and} \qquad c^3 = 64c + 14.$$

Thus,

$$a^3 + b^3 + c^3 = 64(a+b+c) + 3 \times 14 = 64 \times 0 + 3 \times 14 = 42.$$

23. **(d)** Multiplying the original equation by $2xy$, we wish to determine the number of *non-zero* integer pairs x, y such that $2x + 2y = xy$. This equation is equivalent to $(x-2)(y-2) = 4$. Thus, $x - 2$ must be plus or minus a divisor of 4 different from -2. Such a choice, say d, for $x - 2$ uniquely determines x (namely, $x = d + 2$) and uniquely determines y (since $y - 2 = 4/d$, we have $y = (4/d) + 2$). The number of choices for $x - 2$ is 5 (they are ± 1, 2, and ± 4). Therefore, the answer is 5.

24. **(a)** It suffices to compute the ratio of the area of ΔAED to the area of ΔECD. Let h denote their common height from vertex D. The conditions in the problem imply $\angle ACD = 30^\circ$ and ΔACD is isosceles. Hence, $\angle CAD = 30^\circ$ and $\angle ADC = 120^\circ$. By the Law of Sines,

$$\frac{AC}{\sqrt{3}/2} = \frac{AC}{\sin 120^\circ} = \frac{CD}{\sin 30^\circ} = \frac{CD}{1/2}$$

so that $AC = CD\sqrt{3}$. Thus,

$$\frac{\text{area of } \Delta AED}{\text{area of } \Delta ECD} = \frac{\frac{1}{2}h(AC - CD)}{\frac{1}{2}hCD} = \frac{CD\sqrt{3} - CD}{CD} = \sqrt{3} - 1.$$

25. **(a)** Observe that when the product

$$\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right)$$

is expanded, one obtains the sum of the reciprocals of all numbers of the form $2^k \times 3^\ell$ where k and ℓ are non-negative integers. This then is precisely the sum asked for in the problem. Each of the factors in the product above is a geometric series, the first equals $1/(1 - (1/2)) = 2$ and the second equals $1/(1 - (1/3)) = 3/2$. Therefore, the answer is $2 \times 3/2 = 3$. (Note that the product of two converging series cannot always be multiplied together term-by-term and rearranged as above; however, if the series are converging geometric series this can be done. In fact, as long as the series are converging series with positive terms, such term-by-term multiplication and rearrangement does not alter the value of the product.)

26. **(b)** Observe that

$$\left(\frac{1+\sqrt{5}}{2}\right)^3 = 2 + \sqrt{5} \qquad \text{and} \qquad \left(\frac{1-\sqrt{5}}{2}\right)^3 = 2 - \sqrt{5}.$$

Hence,

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=\frac{1+\sqrt{5}}{2}+\frac{1-\sqrt{5}}{2}=1.$$

A more direct approach can be given as follows. Set $\alpha=\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$. Clearly, α is real. Noting that $(2+\sqrt{5})(2-\sqrt{5})=-1$, we obtain

$$\begin{aligned}\alpha^3&=\left(\sqrt[3]{2+\sqrt{5}}\right)^3+3\left(\sqrt[3]{2+\sqrt{5}}\right)^2\left(\sqrt[3]{2-\sqrt{5}}\right)+3\left(\sqrt[3]{2+\sqrt{5}}\right)\left(\sqrt[3]{2-\sqrt{5}}\right)^2+\left(\sqrt[3]{2-\sqrt{5}}\right)^3\\&=(2+\sqrt{5})+3\left(\sqrt[3]{2+\sqrt{5}}\right)(-1)+3\left(\sqrt[3]{2-\sqrt{5}}\right)(-1)+(2-\sqrt{5})=4-3\alpha.\end{aligned}$$

Thus, α is a root of x^3+3x-4. Observe that 1 is clearly a root. In particular, $x-1$ is a factor of x^3+3x-4. Since $x^3+3x-4=(x-1)(x^2+x+4)$ and the roots of x^2+x+4 are imaginary (by the quadratic formula), we deduce that $\alpha=1$.

27. **(c)** We consider the 5 points along the diameter and the 7 points along the half-circular arc separately. Two line segments can intersect at an interior point of the semi-circle in each of the following three ways: (i) the line segments have all 4 endpoints on the arc, (ii) each line segment has one endpoint on the diameter and one endpoint on the arc, and (iii) one line segment has both endpoints on the arc and the other has 1 endpoint on the diameter and 1 on the arc. In each case, a unique intersection point is determined by the 4 endpoints of the two line segments (in other words, if one knows these 4 endpoints, then the two line segments can be drawn in exactly one way so as to intersect at an interior point of the semi-circle). In case (i), the 4 endpoints can be selected in $\binom{7}{4}=35$ ways. Thus, (i) gives rise to 35 interior points. In case (ii), the 4 endpoints can be selected in $\binom{7}{2}\times\binom{5}{2}=21\times 10=210$ ways, producing 210 additional interior points. In case (iii), the 4 endpoints can be selected in $\binom{7}{3}\times\binom{5}{1}=35\times 5=175$ ways, producing 175 more interior points. Hence, there are a total of $35+210+175=420$ interior points where line segments intersect.

28. **(b)** For $1\le j\le 4$, let S_j denote the set of positive integers consisting of j digits in base 4. For example, $S_1=\{1,2,3\}$. The numbers in $S_1\cup S_2\cup S_3\cup S_4$ are precisely the numbers from 1 to $255=4^4-1$. Observe that as m varies over the elements in S_j, the value of $s(m)$ varies over products of the form $d_1d_2\cdots d_j$ where each d_j is a base 4 digit (i.e., some number from the set $\{0,1,2,3\}$). We deduce that

$$\sum_{m\in S_j}s(m)=(0+1+2+3)^j=6^j.$$

Thus,

$$\sum_{m=1}^{255}s(m)=6+6^2+6^3+6^4=6\times(1+6+36+216)=6\times 259=1554.$$

29. **(d)** Drop a perpendicular from P to side $\overline{AB}$, and call the intersection point R. Set $AR=x$ and $BR=y$. Similarly, drop a perpendicular from P to side $\overline{AD}$, and call the intersection point S. Set $AS=u$, and $DS=v$. The conditions in the problem imply

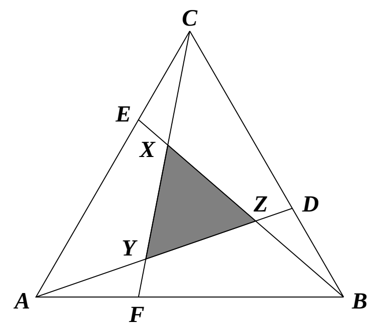
$$y^2+u^2=PB^2=4^2=16,\qquad u^2+x^2=PA^2=8^2=64,\qquad\text{and}\qquad x^2+v^2=PD^2=7^2=49.$$

Hence,

$$PC^2=y^2+v^2=(y^2+u^2)+(x^2+v^2)-(u^2+x^2)=16+49-64=1.$$

Therefore, $PC=1$.

30. **(a)** Let X, Y, and Z be the vertices of the shaded triangle as shown. Observe that ΔCEX, ΔAFY, and ΔBDZ all have the same area, say α. Similarly, quadrilateral $EAYX$, quadrilateral $FBZY$, and quadrilateral $DCXZ$ all have the same area, say β. Let γ denote the area of ΔXYZ. Since ΔCFB and ΔCAF have the same altitude drawn from C and since $FB = 2AF$, we deduce that the area of ΔCFB is twice the area of ΔCAF. Hence, $\alpha + 2\beta + \gamma = 2(2\alpha + \beta)$ which implies $\gamma = 3\alpha$. Now, we see that the areas of ΔAFY and ΔCFA are exactly one third of the areas of ΔXYZ and ΔABC, respectively. Therefore, the ratio of the area of ΔXYZ to the area of ΔABC is equal to the ratio of the area of ΔAFY to the area of ΔCFA. We compute the latter. Observe first that ΔAFY and ΔCFA are similar (since $\angle AFY = \angle CFA$ and $\angle FAY = \angle FCA$). By the Law of Cosines,



$$\begin{aligned} CF^2 &= AC^2 + AF^2 - 2 \times AC \times AF \times \cos\angle CAF \\ &= (3AF)^2 + AF^2 - 2(3AF)AF\cos 60^\circ \\ &= 7AF^2. \end{aligned}$$

Since the ratio of the squares of the lengths of two corresponding sides of similar triangles is the ratio of the areas of the triangles, we deduce that the ratio of the area of ΔAFY to the area of ΔCFA is $1/7$.