High School Math Contest
University of South Carolina
January 15, 2000

1. What is the value of x if $x > 0$ and $72x^2 = 9800$?

   (a) 35/3 (b) 7/4 (c) 100/9 (d) $3\sqrt{10}$ (e) $2\sqrt{30}$

2. What is the value of $\log_2(\log_2(\log_2 16))$?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

3. Which one of the following numbers is smallest?

   (a) 2^{600} (b) 3^{500} (c) 4^{400} (d) 5^{300} (e) 6^{200}

4. Given that $1.00000035811231^2 = 1.000000xyz2247482444265735361$ where x, y, and z denote missing digits, what is the value of $x + y + z$?

   (a) 11 (b) 14 (c) 15 (d) 17 (e) 18

5. What is the value of k in the polynomial identity below?

   $(x^3 - x^2 - 5x - 2)(x^4 + x^3 + kx^2 - 5x + 2) = x^7 - 4x^5 - 14x^4 - 5x^3 + 19x^2 - 4$

   (a) -2 (b) -1 (c) 0 (d) 1 (e) 2

6. Suppose we draw 100 horizontal lines and 100 vertical lines in the plane. How many "pieces" of the plane are formed by cutting along all of these lines? Note: some of the pieces will have infinite area.

   (a) 10000 (b) 10001 (c) 10004 (d) 10201 (e) 10204

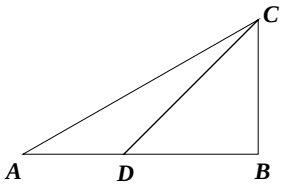
7. Let N be the smallest positive number which is the cube of one integer and the fifth power of a different integer. How many digits does N have?

(a) 3 (b) 4 (c) 5 (d) 8 (e) 15

8. What is the value of $(\sin 15^\circ)^2(\cos 15^\circ)^2$?

(a) 1/2 (b) 1/4 (c) 1/8 (d) 1/16 (e) 1/32

9. Suppose $\angle ABC = 90^\circ$, $\angle CDB = 45^\circ$, $\angle CAB = 30^\circ$, and $AD = 2$. Then BC equals



(a) $\sqrt{3}+1$ (b) $2\sqrt{2}$ (c) $2\sqrt{3}-1$ (d) $\sqrt{6}$ (e) $4-\sqrt{2}$

10. Which integer is nearest in value to the quantity $\dfrac{\sqrt[3]{829}+\log_{10} 829}{2}$?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

11. How many points do the graphs of $4x^2 - 9y^2 = 36$ and $x^2 - 2x + y^2 = 15$ have in common?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

12. Define a sequence by $a_1 = 1$ and for $n \geq 1$,

$$a_{n+1} = \begin{cases} 0 & \text{if } a_n = 0 \text{ and } n \text{ is odd} \\ 2 & \text{if } a_n = 0 \text{ and } n \text{ is even} \\ 1 & \text{if } a_n = 1 \text{ and } n \text{ is odd} \\ 0 & \text{if } a_n = 1 \text{ and } n \text{ is even} \\ 1 & \text{if } a_n = 2 \end{cases}$$

How many of the numbers $a_1, a_2, a_3, \ldots, a_{100}$ are equal to 2?

(a) 12 (b) 16 (c) 20 (d) 24 (e) 28

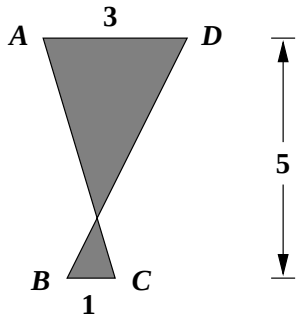
13. How many real numbers are solutions to the equation $x^4 + |x| = 10$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

14. The graph of the equation $x^2 - xy + x - y = 0$ is

(a) an ellipse (b) a parabola (c) a point
(d) a line (e) a pair of intersecting lines

15. Suppose that $\overline{AD}$, $\overline{BC}$, $\overline{AC}$, and $\overline{BD}$ are line segments with line $\overleftrightarrow{AD}$ parallel to line $\overleftrightarrow{BC}$. If $AD = 3$, $BC = 1$, and the distance from $\overleftrightarrow{AD}$ to $\overleftrightarrow{BC}$ is equal to 5, then what is the sum of the areas of the two shaded triangles?



(a) 6 (b) 6.25 (c) 6.5 (d) 6.75 (e) 7

16. What is the coefficient of x^{18} in the polynomial

$$(1+x)^{20} + x(1+x)^{19} + x^2(1+x)^{18} + \cdots + x^{18}(1+x)^2 \ ?$$

(a) 1310 (b) 1320 (c) 1330 (d) 1340 (e) 1350

17. There are four cowboys in a saloon. At midnight, each cowboy randomly chooses one of the other three cowboys and shoots him. What is the probability that exactly two cowboys are shot?

(a) 1/2 (b) 1/3 (c) 1/4 (d) 5/16 (e) 8/27

18. Let $k_1, k_2, \ldots, k_7$, and N be integers such that

$$k_1 + k_2 \times 10 + \cdots + k_7 \times 10^6 = N$$

and

$$k_1 \times 10^6 + k_2 \times 10^5 + \cdots + k_7 = 3N.$$

Which one of the following is a possible value for N?

(a) 41053290 (b) 51053290 (c) 61053290
(d) 71053290 (e) 81053290

19. The following inequalities hold for all positive integers n:

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{4n+1}} < \sqrt{n} - \sqrt{n-1}.$$

What is the greatest integer which is less than

$$\sum_{n=1}^{24} \frac{1}{\sqrt{4n+1}} \ ?$$

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

20. Consider the points $A(-5,-1)$, $B(-1,0)$, $C(1,2)$, and $D(1,3)$. Let P be a point and let $d = PA^2 + PB^2 + PC^2 + PD^2$ so that d is the sum of the squares of the distances from P to each of A, B, C, and D. What is the least possible value for d?

(a) 30 (b) 34 (c) 36 (d) 38 (e) 42

21. How many solutions does the equation $\log_x(5x-2) = 3$ have in real numbers $x > 2/5$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) infinitely many

22. Let a, b, and c be the three roots of $x^3 - 64x - 14$. What is the value of $a^3 + b^3 + c^3$?

(a) -36 (b) 12 (c) 36 (d) 42 (e) 64

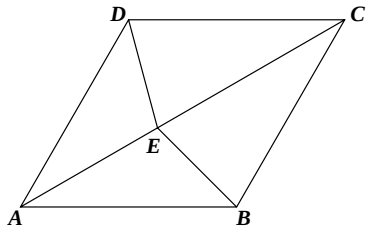
23. How many pairs of integers x, y are there which satisfy the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2} \ ?$$

Note: $x = 1, y = -2$ and $x = -2, y = 1$ are different pairs.

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

24. Consider rhombus $ABCD$ and point E which lies on $\overline{AC}$, the longest diagonal of the rhombus. If $\angle BCD = 60^\circ$ and $CE = CD$, then what is the ratio of the area of quadrilateral $ABED$ to the area of quadrilateral $BCDE$?



(a) $\sqrt{3} - 1$ (b) 2/3 (c) $\sqrt{2}/2$ (d) 3/4 (e) $\sqrt{3}/2$

25. Let S be the set of all positive integers none of whose prime divisors is larger than 3. Thus 1, 2, 3, 4, 6, 8, 9, and 12 are the smallest elements of S. What is the sum of the reciprocals of the elements of S? In other words, what is the value of the sum

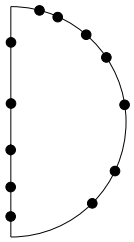
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots \ ?$$

(a) 3 (b) 3.25 (c) 3.5 (d) 3.75 (e) 4

26. The number $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$ equals

(a) $\sqrt{5} - 1$ (b) 1 (c) $\sqrt[3]{2}$ (d) $\sqrt{5} - \sqrt[3]{2}$ (e) 6/5

27. Twelve points are arranged on a semicircle as shown in the diagram. If every pair of these points is joined by a straight-line segment, then no three of these line segments will intersect at a common point inside the semicircle. How many points are there inside the semicircle where two of these line segments intersect?



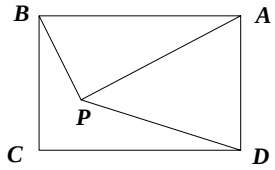
(a) 360 (b) 390 (c) 420 (d) 450 (e) 480

28. For a positive integer n, define $s(n)$ as the product of the base 4 digits of n. For example, since $31 = (133)_4$, we obtain $s(31) = 1 \times 3 \times 3 = 9$. What is the value of

$$s(1) + s(2) + s(3) + \cdots + s(254) + s(255) \ ?$$

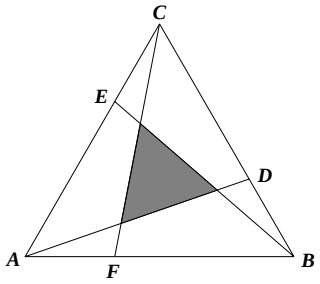
(a) 1496 (b) 1554 (c) 1572 (d) 1596 (e) 1624

29. Let $ABCD$ be a rectangle and let P be a point inside the rectangle. If $PA = 8$, $PB = 4$, and $PD = 7$, then $PC =$



(a) $\sqrt{2}$ (b) 2 (c) $\sqrt{3}$ (d) 1 (e) 3

30. Suppose $\triangle ABC$ is equilateral, $\dfrac{BD}{BC} = \dfrac{1}{3}$, $\dfrac{CE}{CA} = \dfrac{1}{3}$, and $\dfrac{AF}{AB} = \dfrac{1}{3}$. Then the area of the shaded triangle divided by the area of $\triangle ABC$ equals



(a) $\dfrac{1}{7}$ (b) $\dfrac{5}{27}$ (c) $\dfrac{2}{15}$ (d) $\dfrac{3}{20}$ (e) $\dfrac{1}{6}$