MATHEMATICS CONTEST
UNIVERSITY OF SOUTH CAROLINA
3 DECEMBER 1994

INSTRUCTIONS

1. Do not open this booklet until you are told to do so.

2. This is a thirty question multiple choice test. Each question is followed by answers marked a, b, c, d, and e. Only one of these is the correct answer. Drawings on the test are not to scale.

3. Using the #2 pencil given to you, record your answers on the SCANTRON sheet (labelled with your name and school) given to you at registration. *Do not make any marks on this sheet except for your answers.* Use only the first 30 lines of the side labelled with your name. Record your answers with heavy marks, and if you must make corrections, erase thoroughly.

4. The test will be scored as follows: Five points for each correct answer, one point for an answer that is left blank, zero for an incorrect answer. Thus you are discouraged from guessing.

5. You may work the problems in this booklet using it as scratch paper. It is yours to keep after the test.

6. When you are given the signal, begin working the problems. You have 90 minutes working time for the test.

7. If your booklet has a defective, missing, or illegible page, notify the person administering the test.

8. When you have finished the test or the 90 minute time period has elapsed, return the SCANTRON sheet and pencil to the person giving the test.

Mathematics Contest
University of South Carolina
3 December 1994

1. During registration a student discovers that he may register for a Mathematics class at the hours of 8:00, 10:00, or 11:00; English at 9:00, 10:00, or 11:00; and Science at 8:00, 10:00, or 11:00. In how many different ways can he schedule the three courses?

(a) 12 (b) 27 (c) 15 (d) 18 (e) 10

2. Where $i = \sqrt{-1}$, $2i$ is a root of the equation

$$x^4 + 7x^3 + 13x^2 + 28x + 36 = 0.$$

Another of its roots is

(a) $4i$ (b) $\frac{-5+\sqrt{3}}{2}$ (c) $\frac{6+\sqrt{11}}{4}$ (d) $\frac{2+3i}{2}$ (e) $\frac{-7+\sqrt{13}}{2}$

3. $\sin(\pi/12)\sin(5\pi/12) =$

(a) $\sqrt{3}/4$ (b) $\sqrt{2}/2$ (c) $\sqrt{3}/2$ (d) $1/4$
(e) none of (a) through (d)

4. The value of the product

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{99^2}\right)\left(1-\frac{1}{100^2}\right)$$

is

(a) 1/2 (b) .505 (c) 3/5 (d) 2/3 (e) 3/4

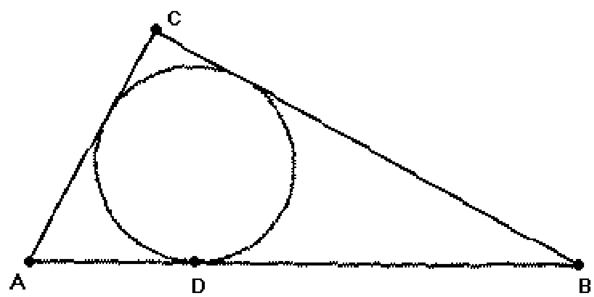
5. Bob starts from the east end and Jane from the west end of a swimming pool, and both swim two lengths of the pool at constant rates. They pass each other twice, each time going in opposite directions. The first time they pass they are 20 feet from the east end, and the second time they are 18 feet from the west end. Assuming that each made an instantaneous turn when they reached an end of the pool, how long is the pool, in feet?

(a) 36 (b) 38 (c) 40 (d) 42 (e) 44

6. $(\log_8 3)(\log_9 4) =$

(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/5 (e) 1/6

7. A circle is inscribed in right triangle ABC, where the right angle is at point C. The circle is tangent to the segment AB at D and the lengths of the segments AD and DB are 7 and 13, respectively. Find the area of the triangle.



(a) 91 (b) 96 (c) 100 (d) 104 (e) 109

8. For each positive integer n, define

$$S_n = 1^4 + 2^4 + 3^4 + \cdots + n^4.$$

What is the value of

$$\log_{10}(S_{100} - S_{99})?$$

(a) 4 (b) 8 (c) 9 (d) 10 (e) 100

9. If (x, y) is a point on the circle $x^2 + y^2 = 1$ and the distance from (x, y) to $(0, 1)$ is $6/5$, then $y =$

(a) 0.28 (b) 1/4 (c) 1/3 (d) 0.24 (e) 1/5

10. The area of the region consisting of those points (x, y) for which $x^2 + y^2 \leq 2$ and $(x-2)^2 + y^2 \leq 2$ is

(a) 2 (b) $\pi - 2$ (c) 3/2 (d) $\frac{\pi}{2} - 1$ (e) 1

11. The numbers $1, 2, 3, \ldots, 100$ are written on 100 cards with one number on each card. The cards are placed into a hat, and one card is selected. The sizes and shapes of the cards are such that the probability of having selected the card labelled with the number n is equal to n times the probability of having selected the card labelled 1. What is the probability that the card labelled 50 was selected?

(a) 1/50 (b) 1/88 (c) 1/101 (d) 2/127 (e) 2/135

12. A tank has three independent inlet pipes, A, B, and C. A and B will fill the tank in z minutes; A and C will fill the tank in y minutes; and B and C will fill the tank in x minutes. How long will it take for pipe A alone to fill it?

(a) $\dfrac{xz + xy - yz}{2xyz}$ (b) $\dfrac{xyz}{2(xz + xy - yz)}$ (c) $\dfrac{2xyz}{xy + yz - xy}$

(d) $\dfrac{2xyz}{xz + xy - yz}$ (e) $\dfrac{xyz}{xy + yz - xy}$

13. A triangle has sides of lengths 1, 2, and $\sqrt{7}$. The measure in radians of the angle opposite the side of length $\sqrt{7}$ is

(a) $\pi/2$ (b) $5\pi/8$ (c) $2\pi/3$ (d) $3\pi/4$ (e) $5\pi/6$

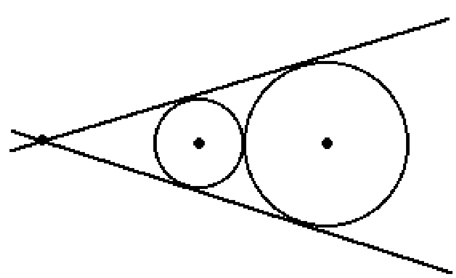
14. The largest integer that is less than $\sqrt{2^{100} + 10^{10}}$ is

(a) 2^{50} (b) $2^{50} + 10$ (c) $2^{50} + 100$ (d) $2^{50} + 1000$
(e) $2^{50} + 10^5 - 1$

15. When expanded, the product $(x+2)(x+3)(x+4)\cdots(x+9)(x+10)$ can be written as $a_9x^9 + a_8x^8 + \cdots + a_1x + a_0$. The value of $a_1 + a_3 + a_5 + a_7 + a_9$ is

(a) $11!$ (b) $11! - 9!$ (c) $27(9!)$ (d) $(11! - 9!)/2$ (e) $(11! + 9!)/2$

16. Two circles, one of radius 8 and one of radius 18, are tangent (i.e., they intersect at exactly one point). There are two lines each of which is tangent to both circles, as shown in the diagram. The distance from the intersection of these lines to the center of the circle with radius 8 is



(a) 16.2 (b) 18.5 (c) 20.8 (d) 22.6 (e) 24.4

17. The value of $\sin 705°$ is

(a) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (b) $\frac{\sqrt{2}-\sqrt{6}}{4}$ (c) $\frac{1-\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}-2}{4}$ (e) $\frac{2+\sqrt{3}}{4}$

18. The 9 numbers $1, 2, 3, \ldots, 9$ are put into a 3×3 array so that each number occurs exactly once. The probability that the sum of the numbers in at least one horizontal row is greater than 21 is

(a) $1/7$ (b) $1/9$ (c) $1/10$ (d) $1/15$ (e) $1/21$

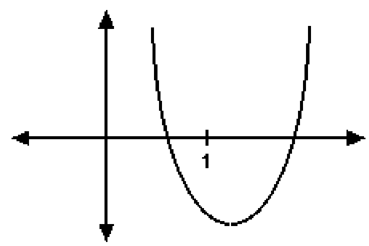
19. A bag contains 5 red marbles and 5 green marbles. One marble is drawn, its color recorded, and then placed back into the bag. This process is repeated until a green marble is found. Given that the first green marble is found on an odd-numbered draw, what is the probability that it is found on the fifth draw?

(a) $1/32$ (b) $3/64$ (c) $3/128$ (d) $1/256$ (e) $3/256$

20. How many incongruent trapezoids $ABCD$ with side AB parallel to side CD have sides of lengths $AB = 16$, $BC = 13$, $CD = 10$, and $DA = 6$?

(a) 0 (b) 1 (c) 2 (d) 4 (e) infinitely many

21. Suppose the graph of $y = ax^2 + bx + c$ is as shown.



Then among the expressions

$$ab, \quad ac, \quad b, \quad a+b+c, \quad a-b+c$$

how many are positive?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

22. For each positive integer n, let

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Then for $n \geq 7$, $nS_n - (S_1 + S_2 + \cdots + S_{n-1}) =$

(a) n (b) $n+1$ (c) $n-1$ (d) $n+2$ (e) $n-2$

23. A point P and a circle C of radius 5 lie in a plane. The shortest distance from P to C is 8. A line passing through P intersects C at exactly one point X; a second line passing through P intersects C at exactly one point Y. What is the distance from X to Y?

(a) $120/13$ (b) 9 (c) 8 (d) $2\sqrt{10}$ (e) $\sqrt{5} + 2\sqrt{2}$

24. Suppose $ABCD$ is a quadrilateral, inscribed in a circle. Then among the following identities:

- $\sin A = \sin C$
- $\sin A + \sin C = 0$
- $\cos B + \cos D = 0$
- $\cos B = \cos D$

how many are always true?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

25. A particle moves inside the square with vertices $A = (0,0)$, $B = (1,0)$, $C = (1,1)$, and $D = (0,1)$. It begins at the point $(2/3, 0)$, travels to some point on the edge $\overline{BC}$, then travels to some point on the edge $\overline{CD}$, then travels to some point on the edge $\overline{DA}$, and then travels to the point $(1/6, 0)$. The minimum distance the particle could have travelled on such a journey is

(a) $\sqrt{6}$ (b) $5/2$ (c) $8/3$ (d) 3 (e) none of (a) through (d)

26. The graph of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ is rotated clockwise around the origin by an angle of $\pi/4$ radians. The equation of the resulting graph can be written in the form $ax^2 + bxy + cy^2 = 72$. The value of a is

(a) 1 (b) -2 (c) 4 (d) 8 (e) 13

27. For the equation $x^2 + 1994x + 3142 = 0$ which of the following is a correct statement?

(a) it does not have any real roots
(b) it has an integer root
(c) it has a positive root
(d) the sum of the reciprocals of the roots is less than -1
(e) none of (a) through (d) is correct

28. Suppose that n is a positive integer having exactly 5 digits and that no two of the 5 digits are the same. Then $n^{28} - n^{26}$ must be divisible by which of the following?

(a) 5 (b) 7 (c) 8 (d) 9 (e) none of (a) through (d)

29. If S is a set of positive integers ≤ 100 with no two distinct elements of S summing to an element of $\{7, 12, 33, 45, 69, 81\}$, then the maximum number of elements S can have is

(a) 56 (b) 60 (c) 64 (d) 65 (e) 68

30. Let S be the set of positive integers that divide at least one of the numbers $1,\ 11,\ 111,\ 1111, \ldots$. For example, 3 is in S since 3 divides 111 . The number of elements in S that are less than 100 is

(a) 22 (b) 28 (c) 34 (d) 40 (e) 48