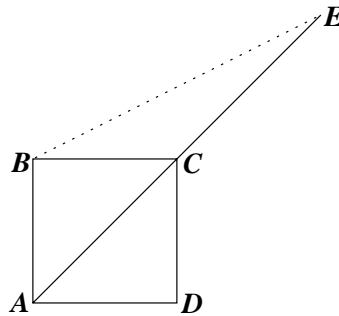


High School Math Contest
University of South Carolina
December 1, 1990

1. Two successive discounts of 10% have the same effect as a single discount of
(a) 18% (b) 19% (c) 20% (d) 21% (e) 22%
2. If $x + 2y = 11$ and $3x + y = 13$, then $x + y =$
(a) 3 (b) 4 (c) 5 (d) 6 (e) 7
3. A pair of fair dice is thrown. What is the probability that the two numbers that appear differ by exactly 2?
(a) $1/6$ (b) $1/3$ (c) $2/3$ (d) $1/9$ (e) $2/9$
4. If $2^a = 5$ and $2^b = 3$, then $\log_3 10 =$
(a) $\frac{a+1}{b}$ (b) $\frac{b+1}{a}$ (c) $\frac{b+1}{a+1}$ (d) $\frac{a}{b}$ (e) $\frac{b}{a}$
5. The area of the square $ABCD$ is 1. Diagonal AC is extended its own length to the point E , so that C is the midpoint of segment AE . Find the length of segment BE .



- (a) $\sqrt{5}$ (b) 2 (c) $\frac{3}{2}\sqrt{2}$ (d) $\sqrt{3}$ (e) $\frac{3}{2}\sqrt{3}$
6. Suppose that $x + \frac{1}{x} = 3$. Then $x^3 + \frac{1}{x^3} =$
(a) 9 (b) 15 (c) 18 (d) 24 (e) 27

7. Find the sum of the infinite series

$$\frac{1}{3} + \frac{2}{3^2} + \frac{1}{3^3} + \frac{2}{3^4} + \frac{1}{3^5} + \frac{2}{3^6} + \frac{1}{3^7} + \cdots$$

- (a) $1/2$ (b) $5/9$ (c) $2/3$ (d) $3/4$ (e) $5/8$

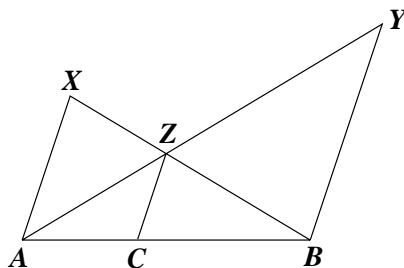
8. For all real numbers x , the function $f(x)$ satisfies

$$2f(x) + f(1 - x) = x^2.$$

Find $f(5)$.

- (a) 4 (b) $34/3$ (c) 8 (d) $19/3$ (e) 12

9. In the diagram below, the line segments AX and BY are parallel to the segment CZ . If the length of AX is 4 and the length of BY is 6, then the length of CZ is



- (a) $3\frac{1}{6}$ (b) $2\frac{1}{3}$ (c) $2\frac{2}{5}$ (d) $3\frac{1}{2}$ (e) $3\frac{1}{8}$

10. If a is a solution to $x^2 - x - 1 = 0$, what is the value of $a^6 - 2a^5 + a^3 + 5$?

- (a) 0 (b) 1 (c) 3 (d) 5 (e) 9

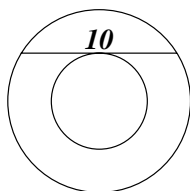
11. Find the number of times between 1 o'clock and 4 o'clock when the hour hand is perpendicular to the minute hand.

- (a) 2 (b) 3 (c) 4 (d) 5 (e) 6

12. Suppose that the squares in the array below are filled with the integers 1 through 16 so that each integer is used just once. Suppose further that every row and every column in the resulting array has the same sum. What is this common sum?

- (a) 33 (b) 34 (c) 35 (d) 36 (e) 37
13. On the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers, an operation $*$ is defined by $a * b = |a - b|$. Which of the following properties is true about the system \mathbb{N} coupled with $*$?
- A. The operation $*$ is commutative.
 B. The operation $*$ is associative.
 C. There is an identity element.
 D. Each element has an inverse.
- (a) only A (b) only C and D (c) only A, B, and C
 (d) only A, C, and D (e) A, B, C, and D
14. Find the number of ordered pairs (x, y) , with x and y both integers, that satisfy the equation $x^2 - 4y^2 = -3$.
- (a) 0 (b) 2 (c) 3 (d) 4 (e) 6
15. Two circular disks, each of radius 1 cm, are centered at points A and B , respectively. The distance between A and B is $\sqrt{2}$ cm. Find the area of the intersection of the two disks, in cm^2 .
- (a) $\frac{\pi}{2} - 1$ (b) $\frac{\pi}{2} - \sqrt{2}$ (c) $\frac{\pi}{4}$ (d) $\sqrt{2}$ (e) $\sqrt{2} - 1$

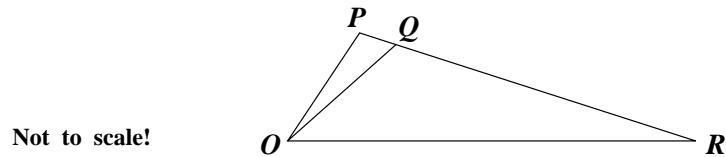
16. A circular lighthouse floor has a circular rug in the center. The lighthouse keeper observes that if he places a 10-foot pole on the floor so that each end touches the wall, then it is tangent to the rug. Find the area of the floor that is not covered by the rug.



- (a) 5π (b) 15π (c) 20π (d) 25π (e) 100π
17. In the expansion
- $$(3x^8 - 2x^6 + x^5 + 2x^4 - x^2 + 1)^5 = a_0 + a_1x + a_2x^2 + \cdots + a_{40}x^{40},$$
- the sum $a_0 + a_2 + a_4 + \cdots + a_{38} + a_{40} =$
- (a) 256 (b) 512 (c) 528 (d) 540 (e) 1024
18. Which equation describes a hyperbola with an asymptote of slope 2?
- (a) $10x^2 - y^2 - 3xy = 10$ (b) $y = 2x^2 + 3$ (c) $x^2 - 2y^2 + 5 = 0$
 (d) $x^2 + 2y^2 - xy = 4$ (e) $x^2 - y^2 + 3x - 2y = 2$
19. Find the value of $\frac{\cos(15^\circ) + \sin(15^\circ)}{\cos(15^\circ) - \sin(15^\circ)}$.
- (a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) 2 (d) 3 (e) $1/3$
20. Suppose that we label each of 8 cards with +1 on one side and -1 on the other. The cards are randomly dealt onto a table, so that either side of each card is equally likely to be facing up. Let x be the absolute value of the sum of the numbers that are facing up. The most likely value of x is
- (a) 0 (b) 2 (c) 4 (d) 6 (e) 8
21. If John gets a 97 on his next math test, his average will be 90. If he gets 73, his average will be 87. How many tests has John already taken?
- (a) 3 (b) 4 (c) 5 (d) 6 (e) 7

22. The largest integer n such that 2^n divides $17^9 - 9^9$ is
- (a) 3 (b) 4 (c) 5 (d) 6 (e) 7

23. In the triangle below, angle POR is 60° . Assume that the line segments OQ and QR have the same length, and line segments OP and PQ have the same length. Find the angle QRO .



- (a) 18° (b) 20° (c) 22.5° (d) 26° (e) 30°
24. Find the sum of the following finite series,

$$\frac{1}{\sqrt{2} + \sqrt{1}} + \frac{1}{\sqrt{3} + \sqrt{2}} + \cdots + \frac{1}{\sqrt{25} + \sqrt{24}},$$

whose n th term is $\frac{1}{\sqrt{n+1} + \sqrt{n}}$.

- (a) 3 (b) 3.8 (c) 4 (d) 4.6 (e) 4.8
25. What is the radius of the circle which circumscribes a triangle with sides of lengths 2, 3, and 4?
- (a) $\frac{8\sqrt{17}}{17}$ (b) $\frac{8\sqrt{15}}{15}$ (c) $\frac{8\sqrt{24}}{24}$ (d) 2.5 (e) 2.25
26. How many permutations of the numbers 1, 2, 3, 4, 5, 6, 7 have 1 somewhere to the left of 2, 2 somewhere to the left of 3, and 3 somewhere to the left of 4?
- (a) 100 (b) 180 (c) 200 (d) 210 (e) 230

27. If r , s , and t are the complex roots of the equation

$$x^3 + 2x^2 + 5x + 7 = 0,$$

then $r^2 + s^2 + t^2 =$

- (a) -6 (b) 6 (c) -4 (d) 4 (e) 2

28. For each real number x , let $f(x)$ be the minimum of the numbers $4x + 1$, $x + 2$, and $-2x + 4$. Then the maximum value of $f(x)$ is
- (a) 2 (b) $5/2$ (c) $8/3$ (d) $17/6$ (e) 3
29. Find the product $\cos(20^\circ) \cos(40^\circ) \cos(80^\circ)$.
- (a) $1/2$ (b) $1/4$ (c) $1/8$ (d) $1/16$ (e) none of these
30. A number $m = 111\dots 11$ consists entirely of 1's in its decimal representation, and is divisible by 13. Which number below need not divide m ?
- (a) 3 (b) 7 (c) 11 (d) 17 (e) 37