MATHEMATICS CONTEST
UNIVERSITY OF SOUTH CAROLINA
DECEMBER 2, 1989

Note: all images are missing from this on-line version of our contest. They will be added at a later date.

1. In the plane diagram below, all indicated angles are right angles. Find the distance from A to B.

   (a) 25 (b) 36 (c) 24 (d) $4\sqrt{2}$ (e) $3\sqrt{3}$

2. What is the binary representation of the fraction 1/5?

   (a) $\overline{.1001}$ (b) $\overline{.0011}$ (c) $\overline{.0101}$ (d) $\overline{.00111}$ (e) $\overline{.01101}$

3. If the system of equations

$$\begin{cases} 2x - 3y &= -4 \\ 3x - y &= 1 \\ x - ky &= 5 \end{cases}$$

   has a solution, then the value of k must be,

   (a) 4 (b) -1 (c) 3 (d) -2 (e) -3

4. Simplify $\dfrac{\cos(5x) + \cos(3x)}{\sin(5x) - \sin(3x)}$.

   (a) $\tan(x)$ (b) $\cot(x)$ (c) $\tan(2x)$ (d) $\cot(2x)$ (e) $\dfrac{\tan(2x)}{\tan(x)}$

5. Let X and Y denote sets. Four of the following five statements imply each other. Which one does not imply any one of the other four?

   (a) $X \cup Y = X$ (b) $X \cap Y = Y$ (c) $X \cap Y \subseteq X \cup Y$
   (d) $Y \subseteq X$ (e) $X \cap Y = X \cup Y$

6. An 8 inch chord is twice as far from the center of a circle as a 10 inch chord. The circumference of the circle in inches is

   (a) $2\sqrt{7}\pi$ (b) $6\sqrt{7}\pi$ (c) $8\sqrt{7}\pi$ (d) $4\sqrt{7}\pi$ (e) $5\sqrt{7}\pi$

7. For what values of p does $4x^2 + 4px + 4 - 3p = 0$ have two distinct real roots?

   (a) $-1 < p < 4$ (b) $-4 < p < 1$ (c) $p < -4$ or $p > 1$
   (d) $p < -1$ or $p > 4$ (e) None of these

8. A rectangle shown below is divided into four rectangles with areas 70, 36, 20, and X. The value of X is

   (a) 350/13 (b) 350/11 (c) 350/9 (d) 350/7 (e) 350/3

9. The fourteen digits on a credit card are to be written in the boxes shown below. If the sum of every three consecutive digits is 18, then the value of X is

   (a) 1 (b) 2 (c) 5 (d) 4 (e) 3

10. The value of $\sin^2(10°) + \sin^2(20°) + \sin^2(30°) + \cdots + \sin^2(90°)$ is

    (a) 5 (b) 6 (c) 4 (d) $3/\sqrt{2}$ (e) $3\sqrt{3}/2$

11. Given a unit square shown below, a square S in the interior is formed by joining each vertex of the unit square to the midpoint of its clockwise nonadjacent side. Then the area of S is

    (a) 1/4 (b) $1/\sqrt{2}$ (c) 1/3 (d) 1/5 (e) 1/2

12. The smallest positive integer N such that the square root of N minus the square root of $N - 1$ is smaller than .01 is

    (a) 2498 (b) 2499 (c) 2500 (d) 2501 (e) 2502

13. If it were two hours later, it would be half as long until midnight as it would be if it were an hour later. The time now is

(a) 9:00 P.M. (b) 7:00 P.M. (c) 8:00 P.M.
(d) 8:30 P.M. (e) 9:30 P.M.

14. If $\sin(x) = 3\cos(x)$, then the value of $\sin(x)\cos(x)$ is

(a) 1/6 (b) 1/5 (c) 2/9 (d) 1/4 (e) 3/10

15. If $4^x - 4^{x-1} = 24$, then the value of $(2x)^x$ is

(a) $4\sqrt{2}$ (b) $12\sqrt{2}$ (c) $10\sqrt{5}$ (d) $4\sqrt{10}$ (e) $25\sqrt{5}$

16. When the base of a triangle is increased by 10% and the altitude to this base is decreased by 10%, the area is

(a) increased by 10% (b) decreased by 10% (c) increased by 1%
(d) decreased by 1% (e) unchanged

17. If the product of the ages of a group of teenagers is 10584000, then the sum of their ages is

(a) 85 (b) 86 (c) 87 (d) 88 (e) 89

18. A "lattice point" is a point in the plane with integer coordinates. How many lattice points are on the line segment whose endpoints are $(3, 17)$ and $(48, 281)$? (Include both endpoints of the segment in your count.)

(a) 2 (b) 4 (c) 6 (d) 16 (e) 46

19. In the figure below, $ABCD$ is a square and ABE is an equilateral triangle. The measure of the angle DCE is

(a) 10° (b) 15° (c) 20° (d) 25° (e) 30°

20. In an English-speaking village of 2029 inhabitants, at least x of the residents have the same two-letter initials. The least possible value of x is

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

21. Suppose that x and y are positive numbers for which $\log_9 x = \log_{12} y = \log_{16}(x+y)$. Then the value of y/x is

(a) $\frac{1}{2}(1+\sqrt{5})$ (b) $\frac{1}{2}(-1+\sqrt{5})$ (c) $\frac{1}{2}(1+\sqrt{3})$
(d) $\frac{1}{2}(-1+\sqrt{3})$ (e) $\frac{1}{2}(2-\sqrt{3})$

22. If $f(x) = ax^2 + bx + c$ and $c = \dfrac{b^2}{4a}$, then the graph of $y = f(x)$ will certainly

(a) have a minimum (b) have a maximum
(c) be tangent to the x-axis (d) be tangent to the y-axis
(e) None of these

23. Given that a, b, and c are the roots of the equation $x^3 - 2x^2 - 11x + 12 = 0$, find $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

(a) 5/6 (b) 11/12 (c) 13/12 (d) 7/6 (e) 8/9

24. If x and y are real numbers and $x^2 + y^2 = 1$, then the maximum value of $(x+y)^2$ is

(a) 3 (b) 1 (c) 2 (d) 3/2 (e) $\sqrt{5}$

25. From a point P outside of a circle with center at O, tangent segments PA and PB are drawn. If $\dfrac{1}{(AO)^2} + \dfrac{1}{(PA)^2} = \dfrac{1}{16}$, then the length of the chord $\overline{AB}$ is

(a) 6 (b) 7 (c) 10 (d) 9 (e) 8

26. The number represented by the infinite sequence $\sqrt{\dfrac{9}{4} + \sqrt{\dfrac{9}{4} + \sqrt{\dfrac{9}{4} + \cdots}}}$ is equal to

(a) $\dfrac{\sqrt{10}-1}{2}$ (b) $\dfrac{\sqrt{10}+1}{2}$ (c) $\dfrac{2\sqrt{2}+1}{3}$
(d) $\dfrac{2\sqrt{2}+1}{2}$ (e) $\dfrac{2\sqrt{2}-1}{2}$

27. If $f(N+1) = (-1)^{N+1}N - 2f(N)$ for integral $N \geq 1$, and $f(1) = f(1989)$, then the value of $f(1) + f(2) + f(3) + \cdots + f(1988)$ is equal to

(a) $-992/3$ (b) $-993/3$ (c) $-996/3$ (d) $-995/3$ (e) $-994/3$

28. The sum of a certain number of positive integers is 31. The largest value that their product can be is

(a) $78,632$ (b) $80,448$ (c) $78,748$ (d) $80,484$ (e) $78,732$

29. An urn contains N black marbles and N white marbles. Three marbles are chosen from the urn randomly and without replacement. What is the value of N if the probability is $1/12$ that all three chosen are white?

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

30. A sequence $\{a_i\}$ is defined as follows:

$$a_{i+1} = \frac{1}{1-a_i} \quad \text{for} \quad i \geq 1.$$

If $a_3 = a_1$, then the value of $(a_9)^9$ is

(a) 2 (b) -2 (c) 1 (d) -1 (e) None of these