High School Math Contest
University of South Carolina
November 12, 1988

1. Sam and Susie are brother and sister. Sam has twice as many sisters as brothers. Susie has twice as many brothers as sisters. The number of girls in the family is

   (a) 2 (b) 3 (c) 4 (d) 5 (e) none of these

2. If greeting cards cost \$2.50 for a box of 12, \$1.25 for a packet of three, or 50 cents each, then the greatest number of cards that can be purchased for \$14.75 is

   (a) 60 (b) 61 (c) 65 (d) 66 (e) 67

3. Points A and B are in the first quadrant and O is the origin. If the slope of $\overline{OA}$ is 1, the slope of $\overline{OB}$ is 7, and the length of $\overline{OA}$ is equal to the length of $\overline{OB}$, then the slope of $\overline{AB}$ is

   (a) $-2/3$ (b) $-3/4$ (c) $-6/7$ (d) $-1/6$ (e) $-1/2$

4. The difference between two numbers is 4. A quotient of the two numbers is -6. The largest possible value for either number is

   (a) $24/7$ (b) $4/7$ (c) $7/4$ (d) 1 (e) arbitrarily large

5. Let $f(x)$ be a quadratic polynomial such that $f(2) = -3$ and $f(-2) = 21$. Then the coefficient of x in $f(x)$ is

   (a) -6 (b) -4 (c) 0 (d) 4 (e) cannot be determined

6. Given that $\log_b\left(a^2\right) = 3$, the value of $\log_a\left(b^2\right)$ is

   (a) $5/3$ (b) $3/4$ (c) $2/3$ (d) $4/3$ (e) $3/2$

7. The integers $2, 3, 4, \ldots$ are arranged in columns as shown below.

A	B	C	D	E
	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	
		$\vdots$		

The column in which 1000 will fall is

(a) A (b) B (c) C (d) D (e) E

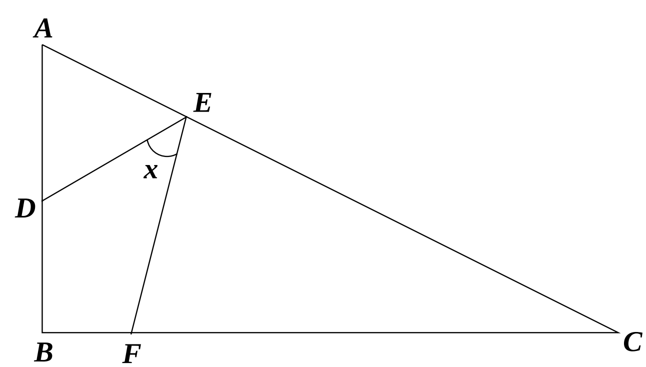
8. Let N be a perfect square. What is the probability that an integer chosen at random from $1, 2, \ldots, N$ will be a perfect square?

(a) $1/N$ (b) $2/N$ (c) $1/(N-1)$ (d) $(N-1)^2/N^2$ (e) $1/\sqrt{N}$

9. Suppose that $a + b = 3$ and $a^2 + b^2 = 7$. Then $a^4 + b^4$ is equal to

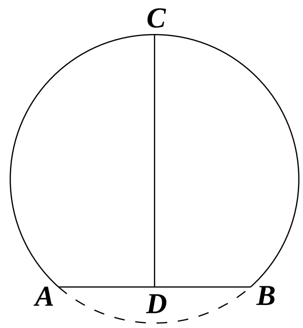
(a) 45 (b) 47 (c) 49 (d) 51 (e) 81

10. In the right triangle ABC below, $AD = AE$ and $CF = CE$. If angle DEF equals $x°$, then x is equal to



(a) 40 (b) 37.5 (c) 47 (d) 45 (e) 42.5

11. In the diagram below, ACB is an arc of a circle, and $\overline{CD}$ is the perpendicular bisector of the chord $\overline{AB}$. If $AD = 3$ and $CD = 9$, then the area of the entire circle is



(a) 16π (b) 20π (c) 25π (d) 24π (e) cannot be determined

12. If $\log_{\sin x}(\cos x) = 1/2$ and $0 < x < \pi/2$, then the value of $\sin x$ is equal to

(a) $\dfrac{1}{\sqrt{3}}$ (b) $\dfrac{\sqrt{5}-1}{2}$ (c) $\dfrac{2}{\sqrt{3}}$ (d) $\dfrac{\sqrt{5}+1}{3}$ (e) $\dfrac{2\sqrt{3}-1}{2}$

13. For the geometric progression $a_1 = 3$, $a_2 = 6$, $a_3 = 12$, $\ldots$, and the aritmetic progression $b_1 = 100$, $b_2 = 400$, $b_3 = 700$, $b_4 = 1000$, $\ldots$, what is the subscript of the first term of the geometric progression that is greater than the corresponding term of the arithmetic progression?

(a) 14 (b) 13 (c) 12 (d) 11 (e) 10

14. A train travelling at 100 kilometers per hour takes three seconds to enter a tunnel and an additional thirty seconds to pass completely through it. The length of the tunnel is

(a) 2/3 km (b) 3/5 km (c) 4/5 km (d) 5/6 km (e) none of these

15. In the binomial expansion of $\left(\dfrac{x^2}{3} - \dfrac{3}{x^2}\right)^6$, the value of the constant term is

(a) 0 (b) -20 (c) -3^6 (d) $\left(\dfrac{1}{3}\right)^6$ (e) $\left(\dfrac{1}{3} - 3\right)^6$

16. $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 1998^2 + 1999^2 =$

(a) $1,000,000$ (b) $1,789,000$ (c) $1,899,000$
(d) $1,989,000$ (e) $1,999,000$

17. The sum

$$\frac{1}{1!9!} + \frac{1}{3!7!} + \frac{1}{5!5!} + \frac{1}{7!3!} + \frac{1}{9!1!}$$

can be written in the form $\dfrac{2^a}{b!}$, where a and b are positive integers. The value of the ordered pair (a, b) is

(a) $(9, 10)$ (b) $(8, 9)$ (c) $(11, 10)$ (d) $(9, 9)$ (e) $(9, 8)$

18. If $|x| + x + y = 10$ and $x + |y| - y = 12$, then $x + y$ is equal to

(a) -2 (b) 2 (c) $22/3$ (d) 22 (e) $18/5$

19. If a and b are integers such that $x^2 - x - 1$ is a factor of $ax^3 + bx^2 + 1$, then b is equal to

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

20. $\sin^4 15^\circ + \cos^4 15^\circ =$

(a) 1 (b) $3/4$ (c) $7/8$ (d) $15/16$ (e) $31/32$

21. The following are statements about trisecting angles using an unmarked straightedge and compass. Only one is true. Which is it?

(a) A construction is known that enables one to trisect any given angle.
(b) Every angle can be trisected, but a construction for doing so has not yet been developed.
(c) An angle can be trisected if and only if its measure is less than 360°.
(d) There are angles that can be trisected, and there are angles that cannot be trisected.
(e) No angle can be trisected.

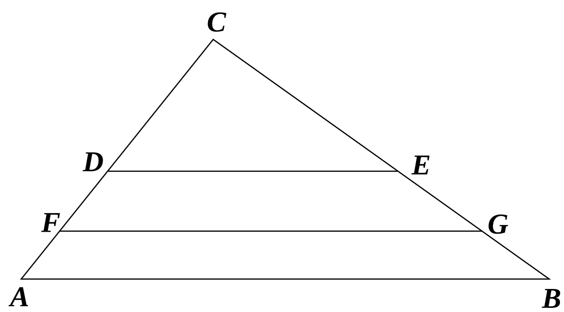
22. Suppose that x and y are positive integers, c is a real number, and i is the imaginary number $\sqrt{-1}$. If $(x+iy)^3 = -74 + ci$, what is the only possible value of the ordered pair (x, y)?

(a) $(2,3)$ (b) $(1,4)$ (c) $(2,4)$ (d) $(1,5)$ (e) $(3,2)$

23. If $x < -2$, then $|1 - |1+x||$ equals

(a) $2+x$ (b) $-2-x$ (c) x (d) $-x$ (e) -2

24. In the figure below, the two lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{FG}$ are both parallel to $\overleftrightarrow{AB}$, and the three regions CDE, $DFGE$, and $FABG$ have equal areas. The ratio of CD to FA is equal to



(a) $\sqrt{3}+1$ (b) $\sqrt{2}+\sqrt{3}$ (c) $\sqrt{5}+1$ (d) $\sqrt{3}+2$ (e) $\dfrac{\sqrt{5}+3}{2}$

25. In the Martian game of QZX, a JBL is worth 7 points and a KMD is worth 4 points. There is no other way to score. Games can continue indefinitely. How many positive integer scores can never be reached? (e.g., no team could ever have a total score of 2 points.)

(a) 5 (b) 7 (c) 9 (d) 11 (e) infinitely many

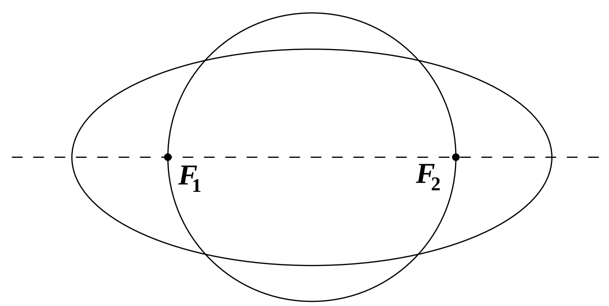
26. A squirrel runs spirally up a cylindrical post, making one circuit for each vertical rise of 4 feet. How many feet does the squirrel travel if the post is 16 feet tall and 3 feet in circumference?

(a) $\sqrt{144^2+16^2}$ (b) $4\sqrt{17}$ (c) 20 (d) 48π (e) $\sqrt{24^2+16^2}$

27. The largest positive integer N such that $\dfrac{(N+1)^2}{N+23}$ is an integer lies in the interval

(a) $[100, 200]$ (b) $[201, 300]$ (c) $[301, 400]$
(d) $[401, 500]$ (e) $[501, 600]$

28. A circle has the same center as an ellipse and passes through the foci F_1 and F_2, as shown below. The two curves intersect at 4 points. Let P be any one of the points of intersection. If the major axis of the ellipse is 15 and the area of the triangle PF_1F_2 is 26, the distance between the foci is



(a) 10 (b) 13 (c) 11 (d) 12 (e) cannot be determined

29. Which of the following are true?

I. $\sin x \leq x$ for $0 \leq x \leq \pi/2$
II. $\sin x \geq \frac{2}{\pi}x$ for $0 \leq x \leq \pi/2$
III. $\sin x \geq \frac{2\sqrt{2}}{\pi}x$ for $0 \leq x \leq \pi/4$
IV. $\sin x \geq \frac{3}{\pi}x$ for $0 \leq x \leq \pi/6$
V. $\sin x \geq \frac{3\sqrt{3}}{2\pi}x$ for $0 \leq x \leq \pi/3$

(a) I only (b) I and II only (c) I, II, and IV only
(d) II, III, IV, and V only (e) all of them are true

30. If x is a complex number such that $x^2 + x + 1 = 0$, then the numerical value of

$$\left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)^2 + \left(x^3+\frac{1}{x^3}\right)^2 + \cdots + \left(x^{27}+\frac{1}{x^{27}}\right)^2$$

is equal to

(a) 52 (b) 56 (c) 54 (d) 58 (e) none of these