High School Math Contest
University of South Carolina
November 8, 1986

1. If $\log_7(x^2) = \log_3(9)$, then the absolute value of x is

   (a) 7 (b) $7/\sqrt{7}$ (c) $7^{1/4}$ (d) $\sqrt{2}$ (e) none of these

2. The sum of the third and fourth terms in a sequence of consecutive integers is 47. The sum of the first five terms of the sequence is

   (a) 90 (b) 72 (c) 115 (d) 93 (e) none of these

3. Given $y > 0$, $x > y$, and $z \neq 0$, the inequality which is not always correct is

   (a) $x - z > y - z$ (b) $xz > yz$ (c) $\frac{x}{z^2} > \frac{y}{z^2}$

   (d) $xz^2 > yz^2$ (e) none of these

4. If Alma wants to mail a package which requires \$1.53 in postage, and has only 5-cent and 8-cent stamps, what is the smallest number of stamps she could use to total exactly \$1.53?

   (a) 24 (b) 23 (c) 21 (d) 14 (e) none of these

5. If $4^x - 4^{x-1} = 24$, then $(2x)^x$ is equal to

   (a) $5\sqrt{5}$ (b) 25 (c) 125 (d) $25\sqrt{5}$ (e) none of these

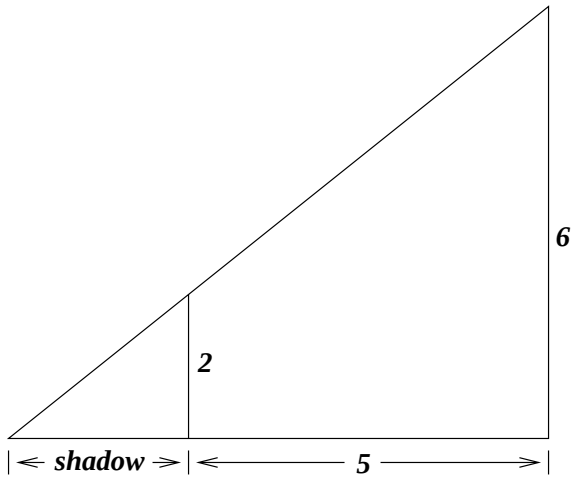
6. If $y = |2\sin(2\pi x - \pi/2) - 1|$, then the largest possible value of y (for real x) is

   (a) 1 (b) 2 (c) 3 (d) 0 (e) none of these

7. Which number below is the greatest?

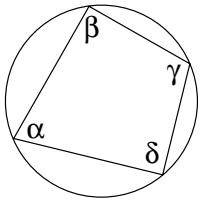
(a) 6^{100} (b) 5^{200} (c) 4^{300} (d) 3^{400} (e) 2^{500}

8. On a dark and lonely night, a man is standing 5 meters away from a street light. The man is 2 meters tall and the light is 6 meters high. How long is the man's shadow?



(a) 2 meters (b) 7/3 meters (c) 5/2 meters
(d) 4 meters (e) none of these

9. The four angles of a quadrilateral inscribed in a circle are α, β, γ, and δ as shown. Which of the following is necessarily true?



(a) $\alpha + \beta + \gamma + \delta = 180^\circ$ (b) $\alpha + \beta = 180^\circ$ (c) $\alpha + \gamma = \beta + \delta$
(d) $\beta + \delta = 90^\circ$ (e) none of these

10. A bus travels up a hill at an average speed of 50 k.p.h. At what average speed would it have to travel down the hill to average 60 k.p.h. for the entire trip?

(a) 68 (b) 70 (c) 72 (d) 73 (e) none of these

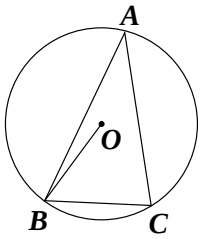
11. The area of the largest triangle that can be inscribed in a semi-circle of radius r is

(a) r^2 (b) $2r^2$ (c) $2r^3$ (d) $\frac{1}{2}r^2$ (e) none of these

12. Expressing the answer only in rational numbers and square roots, the value of $\sec(7\pi/12)$ is

(a) $-\sqrt{2}+3$ (b) $-\sqrt{2}(1+\sqrt{3})$ (c) $\sqrt{2}+3/\sqrt{3}$
(d) $2\sqrt{2}-\sqrt{3}$ (e) none of these

13. In the circle with O as center, $\angle BAC = 15°$. Then $\angle OBC$ is



(a) 30° (b) 75° (c) 45° (d) 60° (e) none of these

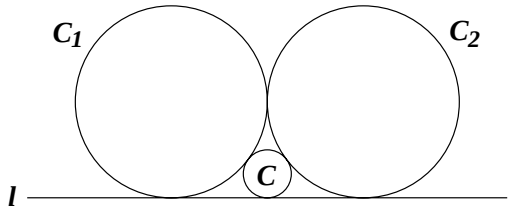
14. A cube measuring 100 units on each side is painted only on the outside and cut into unit cubes. The number of cubes with paint only on two sides is

(a) 1000 (b) 1125 (c) 1176 (d) 980 (e) none of these

15. The largest constant C such that $\sin x \geq Cx$ for all x in $[0, \pi/2]$ is

(a) 0 (b) $1/2$ (c) $2/\pi$ (d) $\pi/4$ (e) none of these

16. Circle C is tangent to line ℓ. Two circles C_1 and C_2 of equal radii are each tangent to one another, to C, and to ℓ. If the radius of C is 3, then the radius of C_1 is

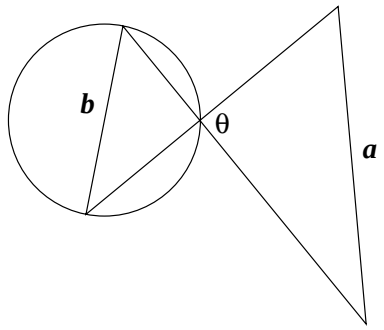


(a) 6 (b) 8 (c) 10 (d) 12 (e) none of these

17. It takes a girl 90 minutes to mow her mother's yard and her brother can do it in 60 minutes. How long would it take them to mow if they worked together using the two mowers?

(a) 36 minutes (b) 150 minutes (c) 40 minutes
(d) 42 minutes (e) none of these

18. In the figure below, given that $\theta = \pi/2$ and that $a/b = 3/2$, the area of the circle in terms of a is



(a) $\frac{\pi a^2}{9}$ (b) $\frac{2\pi a^2}{3}$ (c) $\frac{4\pi a^2}{9}$ (d) $\frac{\pi a^2}{3}$ (e) none of these

19. A lattice point in the plane is a point both of whose coordinates are integers. How many lattice points (including the endpoints) are there on the line segment joining the points $(2, 0)$ and $(16, 203)$?

(a) 15 (b) 8 (c) 9 (d) 14 (e) none of these

20. For what values of a does the system of equations

$$\begin{cases} x^2 &= y^2 \\ (x-a)^2 + y^2 &= 1 \end{cases}$$

have exactly 3 solutions?

(a) for all $a \geq 0$ (b) for $-2 \leq a \leq 2$ (c) for $a = \pm 1$
(d) for $a \in \{-1, 0, 1\}$ (e) none of these

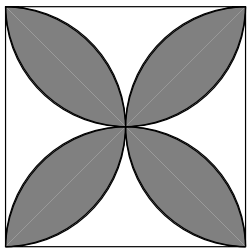
21. Five counterfeit coins are mixed with nine authentic coins. If two coins are drawn at random, the probability that one is good and one is counterfeit is

(a) 1/2 (b) 45/91 (c) 1/14 (d) 4/45 (e) none of these

22. Reading from left to right, the 8th digit of the product $72168482481685664 32 \times 125$ is

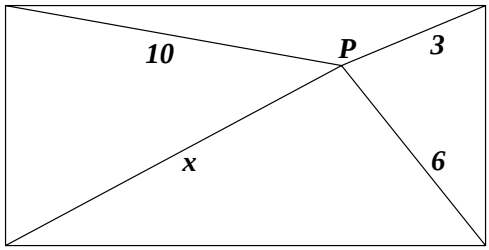
(a) 1 (b) 4 (c) 5 (d) 6 (e) none of these

23. Given a square whose sides have length $2a$, find the area of the region bounded by the 4 semi-circles which are in the interior of the square and have the four sides of the square as diameters.



(a) $(2\pi - 4)a^2$ (b) $\dfrac{\pi a^2}{8}$ (c) $\dfrac{(8-\pi)a^2}{4}$ (d) $2a^2$ (e) none of these

24. In the rectangle below, let p be the point in its interior with the distances from the four corners as shown. The value of x is



(a) 13 (b) $\sqrt{127}$ (c) 7 (d) $\sqrt{109}$ (e) none of these

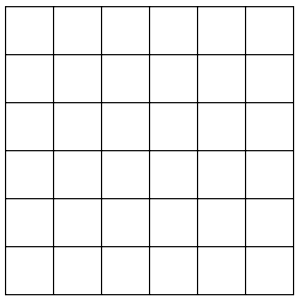
25. A sum of 35 integers is S. Two digits in one of the integers are interchanged and a new sum T is produced. Then the difference $S - T$ is necessarily divisible by

(a) 9 (b) 2 (c) 7 (d) 5 (e) none of these

26. The locus of points equidistant from the circle $x^2 + y^2 = 1$ and the line $y = -3$ is given by the graph of

(a) $y + 2 = 3x^2$ (b) $y + 2 = -3x^2$ (c) $x^2 + y^2 - 1 = (y + 3)^2$
(d) $x^2 + y^2 = (y + 4)^2$ (e) none of these

27. In the configuration below consisting of 36 one-by-one squares, how many total squares with horizontal and vertical sides can be formed using the points as vertices?



(a) 88 (b) 89 (c) 90 (d) 91 (e) none of these

28. The number of solutions, in real numbers x_1, x_2, x_3, to the system of equations

$$\begin{cases} x_1 + x_2 x_3 &= 1 \\ x_2 + x_1 x_3 &= 1 \\ x_3 + x_1 x_2 &= 1 \end{cases}$$

is

(a) zero (b) 3 (c) 4 (d) more than 4, but finitely many
(e) infinitely many

29. Let ℓ_u be the normal line to $y = x^2$ at (u, u^2). Let $(0, b_u)$ be the y-intercept of ℓ_u. Then $\lim_{u \to 0} b_u$ is

(a) 0 (b) 1/2 (c) 1 (d) ∞ (e) none of these

30. $\lim_{n \to \infty} \left(\frac{1}{n} \sum_{k=1}^{n} e^{k/n} \right)$ is

(a) $e - 1$ (b) e (c) $\sqrt{e}$ (d) 1 (e) none of these