Project 2: Koch Snowflake and Fractals

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Overview

The word "fractal" is often used in referring any object that is recursively constructed so that it appears similar at all scales of magnification. There are many examples of complex real-life phenomena, such as chaos, ferns, mountains, river networks, biological growth, that can be described and studied using fractals. In this lab and project, we use the Maple to analyze and to generate a classic fractal, the Koch snowflake, and its variations.

Koch Snowflake

1. **Basic Construction:** It starts with an equilateral triangle. A smaller equilateral triangle is then added to each of the three sides. It is done in such a way that the base of each new triangle is the middle one-third of each side of the original figure. This process is repeated again and again in each successive iteration. Notice that we are getting more and more sides but the basic construction is the same: each side is replaced by four sides of one-third of its length. The Koch snowflake is the limit of this construction.
2. **Working with the Maple:**
   (a) As always, we start a Maple session with `restart` and needed packages.
   ```
   > restart;
   > with(plots):
   ```
   (b) Initial Data: Each side of the initial equilateral triangle is determined by two of its endpoints (vertices) so we need a sequence of four points with the first and the last being the same vertex.
   ```
   > A[1]:=[0,0];
   > A[2]:=[1/2,sqrt(3)/2];
   > A[3]:=[1,0];
   > A[4]:=A[1];
   ```
   (c) Initial Results (level 1): The following should be obvious. Notice that `plot` automatically connects a given sequence of points. Ask your TA if you want to know more about the `sprintf` option in `plot`.
   ```
   > LenSide[1] := 1;
   > TotArea[1] := LenSide[1]*sqrt(3.)/2.*LenSide[1]/2.;
   > NumSide[1] := 3;
   > TotPerim[1] := NumSide[1]*LenSide[1];
   > plot([A[1],A[2],A[3],A[4]],axes=NONE, scaling=constrained, title=sprintf
   ("Koch snowflake (level 1) \n area = % 8.5f, perimeter = % 8.5f, sides =% d",
   TotArea[1], TotPerim[1], NumSide[1]));
   ```
   (d) Results up to level k and limits: Please justify the formulas and limits in your project report.
   ```
   > k:=10:
   > for n from 1 to k-1 do
   >   level:=n+1;
   >   LenSide[n+1] := LenSide[n]/3.;
   >   TotArea[n+1] := TotArea[n]+NumSide[n]*TotArea[1]/9.^n;
   >   NumSide[n+1] := 4*NumSide[n];
   >   TotPerim[n+1] := NumSide[n+1]*LenSide[n+1];
   > end do;
   > TotAreaLimit:=sqrt(3.)*2/5;
   > TotPerimLimit:=infinity;
   ```

(e) Construct sequences of snowflake vertices up to level k: Start from level 1, between each pair of adjacent vertices $A[m]$ and $A[m+1]$, we need to add three new vertices $B[m]$, $E(m)$, and $C[m]$. We then have to re-index them to form a new sequence of vertices at the next level. Pay close attention to your TA's explanation about formulas that produce new vertices, as you need to understand them to complete the project. Also, re-index the vertices explicitly by hand for the first few levels to verify the general formulas. How many vertices are there at level n?

```
> k:=6:        # Computations for levels above 6 may require too much time and
memory for your computer to handle
> for n from 1 to k-1 do
>   for m from 1 to 3*4^(n-1) do
>     B[m]:=A[m]+(A[m+1]-A[m])/3;
># If let A[m+1]-A[m]=[x,y], then [-y,x]=[A[m][2]-A[m+1][2],A[m+1][1]-A[m][1]]
>     E[m]:=A[m]+(A[m+1]-A[m])/2+sqrt(3)/6.*[A[m][2]-A[m+1][2],A[m+1][1]-A[m][1]];
>     C[m]:=A[m]+2*(A[m+1]-A[m])/3;
>   end do;
>   for m from 1 to 3*4^(n-1) do
>     Temp[4*(m-1)+1]:=simplify(A[m]);
>     Temp[4*(m-1)+2]:=simplify(B[m]);
>     Temp[4*(m-1)+3]:=simplify(E[m]);
>     Temp[4*(m-1)+4]:=simplify(C[m]);
>   end do;
>   for m from 1 to 3*4^n do
>     A[m]:=Temp[m];
>   end do;
>   A[3*4^n+1]:=A[1];
># Construct sequence of snowflake plots
>   SFplots[n+1] := seq(plot([A[m],A[m+1]]),m=1..3*4^n):
> end do:
```

(f) Display the fractal growth

```
> for n from 2 to k do
> display([SFplots[n]],axes=NONE, scaling=constrained, title=sprintf
("Koch snowflake (level %a) \n area = %8.5f, perimeter = %8.5f, sides =%d",
n,TotArea[n], TotPerim[n], NumSide[n]));
> end do;
```

Remark: While the for-loop is easy to follow, it is not very efficient in computing. For those who are interested, a link to a faster Maple program for the same construction is included on the lab website.

Instructions: Your report should follow the guidelines set forth in the Project Report Guidelines on the lab website and is due by the date specified by your TA. It should cover the following:

1. Study the given Koch snowflake construction and answer all questions asked. In particular, verify the formulas and limits presented there using your knowledge in sequences and series. Have a discussion on what you have learned, such as some interesting properties of the Koch snowflake. For example, as we have computed, the Koch snowflake has a finite area but infinite perimeter. Now, imaging that you have a container with the Koch snowflake as its base and fill it up with some paint. That means one could paint an infinite area (the interior surface of the container) with a finite amount of paint! The Koch snowflake is also an example of curves that is everywhere continuous but nowhere differentiable.
2. Repeat the same discussion and analysis with your own constructions in Maple of at least two different variations of the Koch snowflake. Use your imagination but one of them must be constructed as follows: Instead of using triangles, start with a square and recursively add smaller squares. Notice that, once you understand the given construction of the Koch snowflake, it should be easily modified for different variations.