# A More Rigorous Approach to Limits

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*Overview*

The rigorous $\epsilon$–$\delta$ definition of limits can be difficult for students to grasp. This lab is designed to provide visual and interactive tools for working with these concepts.

*Maple Essentials*

- The *EpsilonDelta* maplet is available from the course website:

  `http://www.math.sc.edu/calclab/141L-s13/labs/` → EpsilonDelta

*Related course material/Preparation*

§2.4 of the textbook. Let us first recall the definition of limit: Let $f(x)$ be defined for all $x$ in some open interval containing the number $a$, with the possible exception that $f(x)$ need not be defined at $a$. We will write

$$\lim_{x \to a} f(x) = L$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \ \text{ if } \ 0 < |x - a| < \delta.$$

In general, $\epsilon$ and $\delta$ are meant to be very small numbers. Therefore, intuitively, the definition states that $f(x)$ will be very close to $L$ that is, $|f(x) - L| < \epsilon$, when $x$ is very close to $a$ ($|x - a| < \delta$). The task is to show that, for any given $\epsilon$ (no matter how close $f(x)$ is to $L$), you can always find a $\delta$–needed closeness of $x$ to $a$–to make it work.

*Activities*

From our discussion, our job is to find a $\delta$ for a given $\epsilon$ such that, when $a - \delta < x < a + \delta$, the inequality $|f(x) - L| < \epsilon$ holds. Therefore, we need to solve for a range $a - \delta < x < a + \delta$ of $x$ from the given inequality $|f(x) - L| < \epsilon$. Ideally, we would like to find a formula of $\delta$ in terms of $\epsilon$ (see examples 2 and 3 of §2.4) that will work for any given $\epsilon$. However, such formulas are in general very hard to find. Moreover, the value of $\delta$ is not unique, as any value that is smaller than a solution would work, too. For each of the limits below, we will use Maple's `solve` command to help us to find the largest $\delta$ that works for the given $\epsilon$ and the interactive *EpsilonDelta* maplet provides a tool to visualize relations between $\delta$ and $\epsilon$.

(Follow the General Directions on the back of this page.)

1. $\lim_{x \to 9} \sqrt{x} = 3, \ \epsilon = 0.15, \ \epsilon = 0.05$

2. $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} = 6, \ \epsilon = 0.2, \ \epsilon = 0.05$

3. $\lim_{x \to 3} (5x - 2) = 13, \ \epsilon = .05, \ \epsilon = .01$

4. $\lim\limits_{x \to 2}(x^2 + 3x - 1) = 9$, $\epsilon = 0.8$, $\epsilon = 0.6$

   HINT: Since we also have $\lim\limits_{x \to -5}(x^2 + 3x - 1) = 9$, Maple's `solve` command will return extra solutions. Which interval should you choose for problem 4?

## Use Maple's `solve` command to solve inequalities

Maple's `solve` command was introduced in the Lab 4 to solve equations. It can also be used to solve inequalities. We will input most of our inequalities as follows:

> solve(abs(f(x)−L) < ϵ, x);

For example, if we want to know where $|\sqrt{x} - 2| < 0.05$ we would use the following command

> solve(abs(sqrt(x)−2) < 0.05, x);

and Maple would return the interval $(3.8025, 4.2025)$ as the solution (your TA will explain Maple's notation)

## General Directions

1. Look at the limit and identify $f(x)$, $L$, $a$, and $\epsilon$.

2. Launch the *EpsilonDelta* maplet and click **Modify or Make Your Own Problem**. Enter the function $f(x)$, $a$, $L$, and $\epsilon$.

3. Click **Save Problem and Close**. You should see the graph of $f(x)$ in blue with a cyan vertical stripe that goes from $a - \delta$ to $a + \delta$ and a pink horizontal stripe that goes from $L - \epsilon$ to $L + \epsilon$. You should also see a brown rectangle extends vertically from the smallest value of $f(x)$ to the largest value of $f(x)$ for $x$ from $a - \delta$ to $a + \delta$. You may change the size of this rectangle by changing the value of $\delta$, which can be done using the slider (for $0.1 \le \delta \le 1$) or by typing in (any value).

4. Your task is to determine the largest value of $\delta$ that keeps the brown rectangle completely inside the pink stripe. You can use **Zoom In** to increase the accuracy.

5. When you think you are done, record your final value of $\delta$.

6. Now we will find the value of $\delta$ more precisely using Maple's `solve` command.

7. Use the arrow notation (`:=x->`) to define the function $f(x)$. Use `:=` to assign $L$, $a$, and `epsilon` to their respective values.

8. Use the `solve` command as follows

   > solve(abs(f(x) − L) < epsilon, x);

   Maple should return an interval or intervals.

9. Choose the interval that contains $a$. Find the distances from $a$ to the left bound and from $a$ to the right bound of the interval (both of them should be positive.) The *smallest* of these two values is the *largest* $\delta$ that works for this $\epsilon$.

10. Your values from the *EpsilonDelta* maplet and from using the `solve` command should be very close.

## Remark:

For some simple functions like linear functions, `solve` can be used to find general formulas of $\delta$ in term of $\epsilon$. Try the following and compare it to problem 3:

$$\text{solve(abs(5*x-2-13)<epsilon,x) assuming epsilon>0;}$$

## Assignment

Review Lab 1 to Lab 5 for Lab Quiz 1 next week and your lab instructor may give other assignment for each section.