A More Rigorous Approach to Limits

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Overview

The rigorous ϵ–δ definition of limits can be difficult for students to grasp. This lab is designed to provide visual and interactive tools for working with these concepts.

Maple Essentials

- The *EpsilonDelta* maplet is available from the course website:

  `http://www.math.sc.edu/calclab/141L-S08/labs/` → EpsilonDelta

Preparation

Review the precise definition of the limit (pages 138–142 in Anton).
Definition: Let $f(x)$ be defined for all x in some interval containing the number a, with the possible exception that $f(x)$ need not be defined at a. We will write

$$\lim_{x\to a} f(x) = L$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ if } 0 < |x - a| < \delta.$$

In general, ϵ and δ are meant to be very small numbers. Therefore, intuitively, the definition states that $f(x)$ will be very close to L when x is very close to a. The task is to show that, for any given ϵ (no matter how close $f(x)$ is to L), you can always find a δ so that x is close enough to a to make the definition work.

Maple Syntax

For precise solutions to our inequalities, we will be using Maple's `solve` command. The general syntax is

```
> solve(eqn, var);
```

where *eqn* is the equation (or inequality) and *var* is the variable for which we want to solve. We will input most of our inequalities as follows

```
> solve(abs(f(x)-L) < epsilon, x);
```

For example, if we want to know where $|\sqrt{x} - 2| < 0.05$ we would use the following command

```
> solve(abs(sqrt(x)-2) < 0.05, x);
```

and Maple would return the interval $(3.8025, 4.2025)$.

Activities

When using the $\epsilon - \delta$ definition of the limit, we want to find the largest δ that satisfies the definition. For each of the limits below, your task is to identify the δ for each ϵ given.
(Follow the General Directions below.)

1. $\lim_{x\to 9} \sqrt{x} = 3,\ \epsilon = 0.15,\ \epsilon = 0.05$
2. $\lim_{x\to 3} \dfrac{x^2 - 9}{x - 3} = 6,\ \epsilon = 0.2,\ \epsilon = 0.05$ (Page 141, Exercise 11)
3. $\lim_{x\to 3}(5x - 2) = 13,\ \epsilon = 0.10,\ \epsilon = 0.05$ (Page 141, Exercise 10)
4. $\lim_{x\to 2}(x^2 + 3x - 1) = 9,\ \epsilon = 0.8,\ \epsilon = 0.6$
   Hint: For this one, you should use the interval that contains a.

General Directions

1. Look at the limit and identify $f(x)$, a, L, and ϵ.
2. Launch the *EpsilonDelta* maplet and click **Modify or Make Your Own Problem.** Enter the function $f(x)$, a, and L. Enter ϵ.
3. Click **Save Problem and Close**. You should see the graph of $f(x)$ in blue with blue shading that goes from $a - \delta$ to $a + \delta$ along the x-axis. You will notice two red horizontal lines, one at $L - \epsilon$ and the other at $L + \epsilon$. You should also see a brown rectangle that extends vertically from $f(a - \delta)$ to $f(a + \delta)$. You may change the size of this rectangle by changing the value of δ, which can be done using the slider or by typing in the desired value.
4. Your task is to determine the largest value of δ that keeps the brown rectangle completely inside the red lines. You should zoom several times to insure that you have not crossed either horizontal line.
5. When you think you are done, write down your last value of δ that did not cross the line.
6. Now we will find the value of δ more precisely.
7. Use the arrow notation (`:= x ->`) to assign the function $f(x)$. Use `:=` to assign a, L, and `epsilon` to their respective values.
8. Use the `solve` command as follows
   ```
   > solve(abs(f(x) - L) < epsilon, x);
   ```
   Maple will return an interval (or intervals).
9. Find the distances from a to the left bound and from a to the right bound of the interval. (Remember you should use absolute value so both distances are positive.) The *smallest* of these two values is the *largest* δ that works for this ϵ.
10. Your values from the *EpsilonDelta* maplet and from using the `solve` command should be very close.

Remark

Ideally, we would like to find a formula for δ in terms of ϵ (see examples 1, 2, and 3 of §2.4) that will work for any given ϵ. However, such formulas in general are very hard to find. For some simple functions (like linear functions), the `solve` command can be used to find general formulas for δ in terms of ϵ. Try the following and compare the answer with problem 3 above.

```
> restart;
> solve(abs((5*x-2)-13) < epsilon, x) assuming epsilon > 0;
```

Assignment

Exercises 9, 12, and 14 in §2.4 on page 141.
Review Labs A-E for next week's Hour Quiz 1.