

Limits, Infinity, and Asymptotes

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Overview

The concept of a limit is an idea that all other calculus concepts are based on. Limits involving infinity are closely related to asymptotes. While asymptotes for functions are sometimes easy to identify from a graph, the actual definitions of asymptotes is in terms of limits. There are many different types of asymptotes, and the two simplest are:

Asymptote	Equation	Definition
Horizontal	$y = L$	$\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$
Vertical	$x = a$	$\lim_{x \rightarrow a^+} f(x) = \infty$ (or $-\infty$) or $\lim_{x \rightarrow a^-} f(x) = \infty$ (or $-\infty$)

This lab is designed to provide experience finding asymptotes. Some limits will be evaluated by inspection. You will learn several ways to use Maple to help evaluate more difficult limits.

Maple Essentials

- The *Left and Right Limits and Continuity* maplets are available from the course website:

<http://www.math.sc.edu/calclab/141L-F10/labs/> → Using a Graph.
→ Using a Formula.

Note: $\lim_{x \nearrow a} f(x)$ is the limit from below (left) and $\lim_{x \searrow a} f(x)$ is the limit from above (right).

- New Maple commands introduced in this lab include:

Command	Description
<code>limit</code>	finds one- and two-sided limits: <code>limit(f(x), x=a);</code> returns the two-sided limit, $\lim_{x \rightarrow a} f(x)$ <code>limit(f(x), x=a, right);</code> the one-sided limit from above, $\lim_{x \rightarrow a^+} f(x)$ <code>limit(f(x), x=a, left);</code> the one-sided limit from below, $\lim_{x \rightarrow a^-} f(x)$
<code>solve</code>	<code>solve(denom(f(x))=0, x);</code> returns the undefined values of $f(x)$

Preparation

Sections 2.1, 2.3, and 2.6 of Stewart

Activities

- Use the **Left and Right Limits and Continuity** maplets to practice your skills identifying the limits and continuity (first using a graph and then using a formula) of a given function.
 - Launch the first maplet from the course web page and click on **New Function**. Using the graph, determine the limit from the left, the limit from the right, and the value of the function. Notice that these values are independent of one another. That is, they may or may not be equal. Next, decide whether the limit exists and whether the function is continuous. When you have answered all 7 questions, click **Check** to check your work.
 - Repeat the same steps using the second maplet. For this maplet, you will be answering same questions using a formula.
- Using Maple, evaluate each of the following limits. (Stewart Page 107)

a. $\lim_{x \rightarrow -3} \frac{x^2 - 9}{2x^2 + 7x + 3}$	b. $\lim_{x \rightarrow -2} \frac{x + 2}{x^3 + 8}$	c. $\lim_{x \rightarrow 7} \frac{\sqrt{x + 2} - 3}{x - 7}$
d. $\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x}$	e. $\lim_{x \rightarrow 16} \frac{4 - \sqrt{x}}{16x - x^2}$	f. $\lim_{x \rightarrow 2} \frac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$
- Identify all horizontal and vertical asymptotes for as many of the functions on the back of this page as possible.

General Directions

1. Define your function, say $f(x)$, using `f:=x->` and verify that it is entered correctly.
2. Look at the function $f(x)$ and determine which values make the denominator zero. These values will be the a 's that we need to check as possible vertical asymptotes. If you need help, you can use the `solve` command as follows:
`> solve(denom(f(x))=0,x);`
3. To find the vertical asymptotes, you need to evaluate the one-sided limits at $x = a$. Enter the following lines of code:
 (a) `limit(f(x), x=a, left);`
 (b) `limit(f(x), x=a, right);`
 If either of these returns the value ∞ or $-\infty$ then $x = a$ is the equation of a vertical asymptote of $f(x)$.
4. To find the horizontal asymptotes of $f(x)$, you need to find the limits at ∞ and $-\infty$. Enter the following lines of code:
 (a) `limit(f(x), x=infinity);`
 (b) `limit(f(x), x=-infinity);`
 If either of these returns a value $L \neq \pm\infty$ then $y = L$ is the equation of a horizontal asymptote of $f(x)$.
5. Plot the function on a standard window and make sure your answers are consistent with the graph. Use the following line of code:
`> plot(f(x), x=-10..10, y=-10..10, discontinuous);`

Functions

- | | | |
|--|---|---|
| a. $f(x) = \frac{5+2x}{x+1}$ | b. $f(x) = \frac{3x+2}{(x+2)^2}$ | c. $f(x) = \frac{(x^2-1)\sqrt{4x^2+1}}{x^3-2x^2-x+2}$ |
| d. $f(x) = \frac{\sin x}{x}$ | e. $f(x) = \frac{x^3+3x^2-12x+4}{x^2}$ | f. $f(x) = \frac{x^3+3x^2-12x+4}{x^3-x^2-4x+4}$ |
| g. $f(x) = \frac{\sqrt{x^2+1}+2x}{2x-3}$ | h. $f(x) = \frac{\sqrt{9x^2+4}-2}{x+1}$ | |

Assignment

For your assignment this week, complete the following. Turn in a sheet of paper with your answers and attach your Maple worksheet as supporting work.

1. Complete Exercise 8 on page 97 (of Stewart) and exercises 24-30 even on page 107 (of Stewart).
2. Answer each of the following questions. Explain all of your answers and use examples from today's lab where appropriate.
 - (a) Does every undefined value of $f(x)$ lead to a vertical asymptote?
 - (b) Can the graph of a function cross the graph of its horizontal asymptotes?
 - (c) Can the graph of a function cross the graph of its vertical asymptotes?
 - (d) How many horizontal asymptotes can the graph of a function have?
3. Identify all horizontal and vertical asymptotes for the following functions.
 - (a) $f(x) = \frac{6x^2-10x-4}{9x^3+27x^2-x-3}$
 - (b) $f(x) = \frac{\sqrt{16x^2+3}-8x}{4x+1}$