

# Mathematical Models: Designing a Roller Coaster

Douglas Meade, Ronda Sanders, and Xian Wu

Department of Mathematics

## Overview

There are three objectives in this lab:

- understand the mathematical reasoning associated with a real-world example,
- learn to define a piecewise-defined function in Maple, and
- learn to set up and solve a system of equations in Maple.

## Maple Essentials

Important Maple commands introduced in this lab are:

Command	Description
<code>assign</code>	assigns a set of values <code>assign(values);</code>
<code>piecewise</code>	define a piecewise-defined function  The general syntax to represent $\begin{cases} f_1, & cond_1 \\ f_2, & cond_2 \\ \vdots & \vdots \\ f_n, & cond_n \end{cases}$ is: <code>piecewise( cond<sub>1</sub>, f<sub>1</sub>, cond<sub>2</sub>, f<sub>2</sub>, ..., cond<sub>n</sub>, f<sub>n</sub> );</code> where each $cond_i$ is an inequality and each $f_i$ is an expression.
<code>diff</code>	<code>diff(f(x), x);</code> finds the derivative of $f(x)$ with respect to $x$ .

Maple does not recognize double inequalities, so if your condition is  $a \leq x < b$  you would write `x>=a and x<b`.

## Preparation

Review properties of the first derivative.

## Assignment

This week's assignment is to design a larger roller coaster that meets given specifications and prepare a neat and complete project report. **Project 1 will be due at the beginning of the next lab.**

### The Problem: Design a Roller Coaster

Suppose we are asked to design a simple ascent and drop roller coaster with an overall horizontal displacement of 200 feet. By studying pictures of our favorite roller coasters, we decide to create our roller coaster using a line, a parabola and a cubic. We begin the ascent along a line  $y = f_1(x)$  of slope  $\frac{3}{2}$  for the first 20ft horizontally. We continue the ascent and begin the drop along a parabola  $y = f_2(x) = ax^2 + bx + c$  for the next 100ft horizontally. Finally, we begin a soft landing at 30ft above the ground along a cubic  $y = f_3(x) = dx^3 + ex^2 + fx + g$  for the last 80ft.

Here are our tasks:

1. Find a system of 7 equations with the 7 unknowns ( $\{a,b,c,d,e,f,g\}$ ) that will ensure that the track is smooth at transition points.
2. Solve the equations in (1) to find our functions. (We should get a unique solution as we have the same number of equations and unknowns.)
3. Plot the graph to see the design.
4. Find the maximum height of the roller coaster.

### Solving the Problem

1. Start your Maple session with  
`> restart;`  
 This clears the internal memory so that Maple acts (almost) as if just started and is very helpful if you make a mistake and want to start over.
2. We begin by defining our functions in Maple. If we choose the origin as our starting point, our first function  $y = f_1(x)$  is a line of slope  $\frac{3}{2}$  that passes through  $(0,0)$ , and we have:  
`> f1:=x-> 3/2*x;`  
`> f2:=x-> a*x^2+b*x+c;`  
`> f3:=x-> d*x^3+e*x^2+f*x+g;`

3. We will also need the first derivatives of our functions. (If you do not see why, you will soon.) To find and assign the derivatives, right-click over the function and choose **differentiate**. Then right-click over the derivative function and choose **assign to a name**. Name the derivatives  $df_1$ ,  $df_2$ , and  $df_3$ , respectively.
4. Since our roller coaster consists of three curves, it can be set up mathematically as a piecewise-defined function:

$$F(x) = \begin{cases} f_1(x), & 0 \leq x \leq 20 \\ f_2(x), & 20 < x < 120 \\ f_3(x), & 120 \leq x \leq 200 \end{cases}$$

We assign  $F$  as a function in Maple as follows:

```
> F:= x -> piecewise(x<=20, f1(x), x>20 and x<120, f2(x), x>=120 and x<=200, f3(x));
```

**Note:** You can verify your piecewise-defined function by typing  $F(x)$ ;

5. Obviously, we want  $F(x)$  to be continuous (so our passengers do not perish). This means that our functions should be equal at transition points. So we get the following equations:  
`> eq1:=f1(20)=f2(20);`  
`> eq2:=f2(120)=f3(120);`
6. If we are to have a smooth track, we cannot have abrupt changes in direction, so the first derivative  $F'(x)$  should also be continuous. That is, the first derivatives of our functions should also be equal at transition points. So we get:  
`> eq3:=df1(20)=df2(20);`  
`> eq4:=df2(120)=df3(120);`
7. To start our landing at 30ft above the ground for the last 80ft, we would have:  
`> eq5:=f3(120)=30;`
8. Finally, in order to have a soft landing, the track should be tangent to the ground at the end:  
`> eq6:=f3(200)=0;`  
`> eq7:=df3(200)=0;`
9. We now have a system of 7 equations and 7 unknowns. We solve using the **solve** command and assign the solutions as follows:  
`> values:=solve({eq1,eq2,eq3,eq4,eq5,eq6,eq7} , {a,b,c,d,e,f,g});`  
`> assign(values);`
10. You can view your completed piecewise-defined function by typing  
`> F(x);`  
**Note:** If I were preparing a project report about this roller coaster, I would definitely include this function.
11. We can see what our coaster looks like with the following **plot** command:  
`> plot(F(x), x=0..200, y=-50..150);`  
**Note:** Notice that we want to choose the same scale for  $x$  and  $y$ .
12. To find the maximum height, we need to find where the graph has a horizontal tangent line (where  $F'(x) = 0$ ) and evaluate  $F(x)$  at each point. The largest is the maximum height of the coaster.  
`> diff(F(x),x);` **Note:** You can get this by right-clicking over  $F(x)$  above also.  
`> dF:= x -> label;`  
`> solve(dF(x)=0, x);`  
**Note:** We know that  $F'(x) = 0$  when we have a horizontal tangent line (a slope of 0). This occurs at both local maximums and minimums. More detailed discussion will be given in Chapter 5.