Riemann Sums and Integration

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Overview

This lab will help to develop your understanding of the definite integral as defined and computed via Riemann sums and its connection to the indefinite integral (antiderivative) via the Fundamental Theorem of Calculus.

Maple Essentials

- New Maple commands introduced in this lab include:

Command	Description
`int`	used for definite and indefinite integrals
`int(f(x), x);`	evaluates $\int f(x)dx$ **Remember:** Maple will not add +C for you.
`int(f(x), x=a..b);`	evaluates $\int_a^b f(x)dx$

- The *Riemann Sums* tutor can be started from the Tools menu:

  Tools → Tutors → Calculus - Single Variable → Riemann Sums ...

- The *Anti-Derivative Drill* and *Integration by Substitution* maplets provide integration practice. These maplets are available from the course website:

  `http://www.math.sc.edu/calclab/141L-F08/labs/` → AntiDerivativeDrill
  `http://www.math.sc.edu/calclab/141L-F08/labs/` → IntBySub

Preparation

§6.1, §6.4, §6.5, and §6.6 of the textbook.
The definite integral of $f(x)$ is defined as the limit of Riemann sums

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\Delta x.$$

To use the above definition/formula to compute or estimate $\int_a^b f(x)\ dx$, you first choose n (the number of subintervals) and set $\Delta x = (b-a)/n$ (the length of each subinterval). Next, you need to choose x_k^* in each subinterval. Some popular choices are the left endpoint, the right endpoint, or the midpoint of each subinterval. You then increase n to get better and better approximations. Of course, this leads to messy computations, as there are n terms in the sum and a closed form is in general very hard to find. The `Riemann Sums` tutor is a great tool to carry out those computations. It also let you visualize basic ideas behind the definition.

A completely different way to compute definite integrals is to use the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\ dx = F(b) - F(a), \qquad \text{where } F(x) \text{ is an antiderivative of } f(x).$$

The FTC relates definite integrals (which are numbers as signed areas) to indefinite integrals (which are functions as antiderivatives). This is great if you know how to find $F(x)$. The problem is that, as you likely have learned already, it can be very difficult (or even impossible) to find a closed form of $F(x) = \int f(x)dx$. Maple is very capable of finding indefinite integrals but don't be surprised when it fails. Just remember that you can always use Riemann sums to find definite integrals numerically.

Activities

1. Use the `Riemann Sums` tutor to approximate $\int_2^{10} \frac{1}{x}\, dx$ with the Riemann sum $\sum_{k=1}^{4} f(x_k^*)\Delta x$ where:

   (a) x_k^* is the left endpoint of each subinterval
   (b) x_k^* is the right endpoint of each subinterval
   (c) x_k^* is the midpoint of each subinterval

   Then increase the number of subintervals and describe what happens to the approximation.

   Directions:

   (a) Launch the *Riemann Sums* tutor.
   (b) Plug in $f(x) = 1/x$, $a = 2$, $b = 10$, and $n = 4$.
   (c) Click on **left** and press **Display**. Notice how each rectangle has the height of the left endpoint's function value.
   (d) Repeat for **right** and **midpoint**.
   (e) Input other values for n, say 8, 64, 200, etc, clicking **Display** each time. What happens to your approximation?

2. Use Maple to evaluate $\int_2^{10} \frac{1}{x}\, dx$ via the FTC and compare it to the results from *Activity* 1.

   ```
   > f := x-> 1/x;                    # Enter the integrand.
   > int(f(x), x);                    # Find the antiderivative of f(x).
   > F := x-> label;                  # Assign the antiderivative as F(x).
   > Area := F(10) - F(2);            # Apply the Fund Thm of Calculus.
   > int(f(x), x=2..10);              # Calculate the definite integral.
   ```

   Notes/Remarks:

   - To obtain results in decimal, change integral limits, say, 2 and 10, to 2.0 and 10.0.
   - As it has been pointed out, one may not be able to find a closed form of $F(x) = \int f(x)\, dx$. Try the example of $f(x) = (\ln x)e^{-x^2}$ as follows:
     ```
     > int(ln(x)*exp(-x^2), x);
     > int(ln(x)*exp(-x^2), x=2..10);
     ```
     As you can see, Maple did not find a closed form of the indefinite integral and hence failed to evaluate the definite integral via the FTC.
   - However, if you type in floating-point numbers as the integral limits in `int` command, then it will evaluate the integral via Riemann sums instead of the FTC. Try the same example but change integral limits to floating-point numbers as follows:
     ```
     > int(ln(x)*exp(-x^2), x=2.0..10.0);
     ```
     This time it should work - an advantage of Riemann sums over the FTC.

3. Repeat *Activities* 1&2 for the following definite integrals:

   $$\int_2^6 x^3\, dx \qquad \int_{-1}^3 e^{-x}\, dx \qquad \int_0^{3\pi/2} \cos(x)\, dx \qquad \int_0^5 \sqrt{x}\, dx \qquad \int_0^4 \frac{x}{x^4 + 1}\, dx$$

4. Use the *Anti-Derivative Drill* maplet to practice your skills finding the anti-derivative.
   - Under the `Functions/Methods` tab add Trigonometric, Exponential, and Logarithmic functions.
   - Click on `New Function` to work a randomly generated example.
   - Click on `Enter Function` to enter a problem from your homework or study materials.
   - **Don't forget +C!**

5. Use the *Integration by Substitution* maplet to practice your skills finding the anti-derivative when you must first rewrite the problem.

Assignment

There is no assignment this week other than to complete the Lab Activities.