Definite Integrals, Riemann Sums, and FTC

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Overview

This lab will develop your understanding of the definite integral as defined and computed via Riemann sums and its connection with the indefinite integral via the Fundamental Theorem of Calculus (FTC).

Maple Essentials

- Important Maple commands introduced in this lab are:

Command	Description	Example
`int(f,x)`	indefinite integral $\int f(x)dx$	`int(k*epx(x)*sin(x),x);` `int(k*exp(x)*sin(x),k);`
`int(f,x=a..b)`	definite integral $\int_a^b f(x)dx$	`int(sin(x),x=0..1);` `int(sin(x),x=0.0..1.0);`

- The *Riemann Sums* tutor can be started from the Tools menu:

  Tools → Tutors → Calculus - Single Variable → Riemann Sums ...

  Note: *The* Riemann Sums *and* Approximate Integrals *tutors are identical.*

Related course material/Preparation

§6.1, §6.4, §6.5, and §6.6 of the textbook.

The definite integral of $f(x)$ is defined as the limit of Riemann sums

$$\int_a^b f(x)\ dx \quad = \quad \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x.$$

To use the above definition/formula to compute or estimate $\int_a^b f(x)\ dx$, you first choose n (the number of subintervals) and set $\Delta x = (b-a)/n$ (the length of each subinterval). Next, you need to choose x_k^* in each subinterval. Some popular choices are the left endpoint, the right endpoint, or the midpoint of each subinterval. You then increase n to get better and better approximations. Of course, this leads to messy computations, as there are n terms in the sum and a closed form is in general very hard to find. The `Riemann Sums` tutor is a great tool to carry out those computations. It also let you visualize basic ideas behind the definition.

A completely different way to compute definite integrals is to use the FTC

$$\int_a^b f(x)\ dx \quad = \quad F(b)-F(a), \qquad \text{where } F(x) \text{ is an antiderivative of } f(x) \text{ over } [a,b].$$

The FTC relates definite integrals (which are numbers as signed areas) to indefinite integrals (which are functions as antiderivatives). This is great if you know how to find $F(x)$. The problem is that, as you likely have learned already, it can be very difficult (or even impossible) to find a closed form of $F(x)=\int f(x)dx$. The maple is very capable of finding indefinite integrals but don't be surprised when it failed. Just remember that you can always use Riemann sums to find definite integrals numerically.

Activities

1. Use the `Riemann Sums` tutor to approximate $\displaystyle\int_2^{10}\frac{1}{x}\ dx$ with the Riemann sum $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x$ where:

   (a) x_k^* is the left endpoint of each subinterval

(b) x_k^* is the right endpoint of each subinterval

(c) x_k^* is the midpoint of each subinterval

Then increase the number of subintervals and describe what happens to your approximation.

Directions:

(a) Launch the *Riemann Sums* tutor.

(b) Plug in $f(x) = 1/x$, $a = 2$, $b = 10$, and $n = 4$.

(c) Click on left and press **Display**. Notice how each rectangle has the height of the left endpoint's function value.

(d) Repeat for right and midpoint.

(e) Input other values for n, say 8, 16, 32, 64, 128, etc, clicking **Display** each time. What happens to your approximation?

2. Use Maple to evaluate $\displaystyle\int_2^{10} \frac{1}{x}\, dx$ via the FTC and compare it to the results from *Activity* 1.

   Step-by-step implementing:

   ```
   > f :=x-> 1/x;                               # define the integrand
   > F := int( f(x), x );                       # find indefinite integral F(x) of f(x)
   > area:= eval(F, x=10) - eval(F, x=2);       # apply the FTC
   ```

   Notes/Remarks:

   - The above step-by-step sequence can be replaced by one maple command:
     `> int(1/x, x=2..10);`
   - To obtain results in decimal,use `evalf` on results or change integral limits, say, 2 and 10, to 2.0 and 10.0.
   - As it has been pointed out, one may not be able to find a close form of $F(x) = \int f(x)\, dx$. Try the example of $f(x) = (\ln x)e^{-x^2}$ as follows:
     `> int(ln(x)*exp(-x^2), x);`
     `> int(ln(x)*exp(-x^2), x=2..10);`
     As you can see, maple did not find a closed form of the indefinite integral and hence failed to evaluate the definite integral via the FTC.
   - However, if you type in floating-point numbers as the integral limits in `int` command, then it will evaluate the integral via Riemann sums instead of the FTC. Try the same example but change integral limits to floating-points numbers as follows:
     `> int(ln(x)*exp(-x^2), x=2.0..10.0);`

     This time it should work, an advantage of Riemann sums over the FTC.

3. Repeat *Activity* 1 and *Activity* 2 for the following definite integrals:

$$\int_0^{\pi/2} \cos(x)\, dx \qquad \int_2^6 x^3\, dx \qquad \int_{-1}^3 e^{-x}\, dx \qquad \int_0^4 \frac{x}{x+1}\, dx \qquad \int_{-4}^4 \frac{x}{x+1}\, dx$$

$$\int_0^{3\pi/2} \cos(x)\, dx \qquad \int_0^5 \sqrt{x}\, dx \qquad \int_{-1}^3 xe^{-x}\, dx \qquad \int_{-4}^{-2} \frac{x}{x+1}\, dx \qquad \int_0^4 \frac{x}{x^4+1}\, dx$$

Assignment

You have just completed the last Math-141 lab. Congratulations! There is no assignment this week but, for some extra credit lab points, please complete the Maple survey and put it in the box marked with Math-141 Lab Survey in LC 411 before you take the final exam. Don't forget to let your TA know that you have completed the survey so he/she can give you the credit.