Algebraic number theory (Spring 2013), Homework 3

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Due Monday, February 10

1. (5 points) Represent 23, $\frac{1}{4}$, -7, and $-\frac{1}{14}$ as 7-adic numbers. Which of them are 7-adic integers?

2. (5 points) Write out a formal proof that there exists an injection $\mathbb{Z}_{(p)} \to \mathbb{Z}_p$.

3. (*7 points) Look up and write out the definition of an *inverse limit* in general, in terms of a universal property. (For example, see the Wikipedia page.) Prove that $\mathbb{Z}_p$ is the inverse limit of the rings $\mathbb{Z}/(p^n)$, under the projection morphisms, according to this definition.

4. (5 points) Represent $\sqrt{6}$ as a 5-adic integer (find the first few 5-adic digits, and prove that you can keep going without quoting Hensel's lemma), and prove that you cannot represent $\sqrt{6}$ as a 7-adic integer.

5. (10 points) Starting from the completion definition of $\mathbb{Q}_p$ (Cauchy sequences mod Cauchy sequences converging to zero), prove the following properties, less sketchily than was done in lecture:

   - $\mathbb{Q}_p$ is a field.
   - $\mathbb{Z}_p$ is a ring, and (p) is the unique maximal ideal.
   - $\mathbb{Q}_p$ and $\mathbb{Z}_p$ possess an absolute value which agrees with the p-adic absolute value on $\mathbb{Q}$ and $\mathbb{Z}$, and are complete with respect to this absolute value.

6. (5 points) Prove that addition or multiplication by any fixed element of $\mathbb{Q}_p$ is (topologically) a homeomorphism from $\mathbb{Q}_p$ to itself.

   If you want to study valuations in general, the adeles, Tate's thesis, etc., please be sure to do this exercise. (Or just convince yourself it's "obvious".)

7. (7 points) $\mathbb{Z}_p$ is *compact.*