ALGEBRAIC NUMBER THEORY – BONUS PROBLEMS

20 points total — This is due to Ravi Vakil.

(1) The game of Chomp is played between two players. Fix nonnegative integers n and m. Cookies are placed in a rectangular array at the points (x, y) where $0 \leq x \leq m$ and $0 \leq y \leq n$. The cookie at $(0, 0)$ is poisoned. Two players alternate moving; a move involves picking a cookie, and eating it and every cookie above and to the right of it. The player who dies loses.

There is a nice proof that unless m and n are both zero, the first player has a winning strategy. Can you prove this? (Hint: use a *strategy-stealing argument.*)

(2) Suppose now that you play Infinite Chomp, where x and y run through all nonnegative integers. Suppose that you and your opponent cooperate, and attempt to make the game last infinitely many turns.

Prove that your attempt will **end in futility**, that the game will end after **finitely many turns**, and that one of you will **die**. ☠☠☠

(3) Generalize the previous problem to a d-dimensional board where cookies are placed on $(\mathbb{Z}^+)^d$. (In some respects, this problem may be easier.)

(4) Prove the folliowing special case of the *Hilbert Basis Theorem*: The polynomial ring $\mathbb{C}[x_1, x_2, \cdots x_n]$ is Noetherian.