T1.1. Valuations and completions.

Def. A <u>valuation</u> of a field K is a function

$$|\cdot| : K \longrightarrow \mathbb{R} \qquad \text{s.t.}$$

(1) $|x| \geq 0$ and $|x| = 0 \longleftrightarrow x = 0$

(2) $|xy| = |x| \cdot |y|$

(3) $|x+y| \leq |x| + |y|$.

Tacitly exclude the trivial valuation $|x| = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

This defines a <u>distance</u> $d(x,y) = |x-y|$
and a topology.

Definitions.

A valuation is <u>discrete</u> if $\{|x| : x \in K\}$ is discrete
(i.e. is a <u>lattice</u> in $\mathbb{R}$)
~~(i.e. is $\mathbb{Z} \cdot a$ for some $a \neq 0$).~~

Two valuations are <u>equivalent</u> if they induce the same topology.

A valuation is <u>archimedean</u> if $|n|$ is unbounded for $n \in \mathbb{N}$
<u>nonarchimedean</u> otherwise.

Examples.

$K = \mathbb{Q}$. The usual absolute value, and p-adic for all p.

Prop. These are all. (not trivial)

$K = \mathbb{Q}(\sqrt{3})$, for which $\mathcal{O}_K$ is a PID.

not quite { If p is a prime of $\mathcal{O}_K$, get a p-adic valuation
$v_p(x) = p^{-n}$ where $x = p^n \cdot \frac{a}{b}$ (a, b coprime to n).

Also have two nonarchimedean valuations.

(1) $\mathbb{Q}(\sqrt{3}) \doteq \mathbb{Q}[x]/(x^2-3) \longleftrightarrow \mathbb{R}$

$1 \longrightarrow 1$

$x \longrightarrow 1.732\ldots \quad \therefore |\sqrt{3}| = 1.732$

T1.2.

$$\mathbb{Q}[x]/(x^2-3) \longrightarrow \mathbb{R}$$
$$x \longmapsto -1.732\ldots$$

again $|\sqrt{3}| = -1.732$.

But $|2-\sqrt{3}|$ equals $3.732\ldots$ or $.267\ldots$ depending on the valuation.

Ex. Suppose K is a NF not a PID.
Then let $\mathfrak{p}$ be a prime ideal.
Define $v_{\mathfrak{p}}(x) = (N\mathfrak{p})^{-n}$ where $(x) = \mathfrak{p}^n \cdot \left(\begin{smallmatrix}\text{ideal coprime}\\ \text{to } \mathfrak{p}\end{smallmatrix}\right)$.

Note that here (x) is a fractional ideal.
(maybe $(x) \not\subset \mathcal{O}_K$)
(x) is a f.g. $\mathcal{O}_K$-submodule of K.
Fractional ideals are invertible
(not obvious, will discuss later)

Ex. If k is a field, look at $k(t)$.
One valuation: Pick $a \in k$. $v_p(f(t)) = e^{-n}$, where $f(t) = (t-a)^n \cdot$ rat'l fn. coprime to $t-a$.

If k is $\mathbb{F}_q$, maybe substitute q for e.
(equivalent. same topology.)

You also have the degree valuation
In $k[t]$, $v_p(f(t)) = e^{\deg(f)}$.

Prop. Two valuations $|\cdot|_1$ and $|\cdot|_2$ are equivalent iff $\exists$ a real number $s>0$ s.t.
$$|x|_1 = |x|_2^s \quad \text{for all } x \in k.$$

T1.3.

Proof. If $|\cdot|_1 = |\cdot|_2^s$ with $s>0$, then obviously equivalent.
Conversely, suppose $|\cdot|_1$ and $|\cdot|_2$ are equivalent.

Then, $|x|<1 \longleftrightarrow \{x^n\}_{n\in\mathbb{N}} \longrightarrow 0$ in $|\cdot|$,

So, $|x|_1 < 1 \longleftrightarrow |x|_2 < 1$.

Note. $|x|_1<1 \longmapsto |x|_2<1$ will be enough.

Now, suppose $y \in K$ is any element with $|y|_1 > 1$.

~~Choose x with $|x|_1 > 1$, and~~

For any x, $x \neq 0$, $|x|_1 = |y|_1^{a}$. (some $a \in \mathbb{R}$)

Let $\frac{m_i}{n_i}$ be a sequence of rat'l numbers $(n_i > 0)$ approaching a from above.

Then $|x|_1 = |y|_1^{a} < |y|_1^{m_i/n_i}$, and

$$\left|\frac{x^{n_i}}{y^{m_i}}\right|_1 < 1 \longmapsto \left|\frac{x^{n_i}}{y^{m_i}}\right|_2 < 1, \text{ so}$$

$|x|_2 \leq |y|_2^{m_i/n_i}$. So $|x|_2 \leq |y|_2^{a}$.

By choosing a sequence $\frac{m_i}{n_i}$ approaching from below,

$|x|_2 \geq |y|_2^{a}$.

So $|x|_2 = |y|_2^{a}$.

So, $$\frac{\log|x|_1}{\log|y|_1} = \frac{\log|x|_2}{\log|y|_2}$$

and so $$\frac{\log|x|_1}{\log|x|_2} = \frac{\log|y|_1}{\log|y|_2} =: s.$$

So $|x|_1 = |x|_2^s$. And $s>0$ because

$|y|_1 > 1 \longrightarrow |y|_2 > 1$.

TI.4. In fact, as we will need, this shows more.

Prop. TFAE.

(1) $|\cdot|_1$ and $|\cdot|_2$ are equivalent

(2) $|x|_1 < 1 \longleftrightarrow |x|_2 < 1$

(3) $|x|_1 < 1 \longmapsto |x|_2 < 1$

(4) $|x|_1 = |x|_2^s$ for some $s > 0$.

Statement of the theorem was (1) $\longleftrightarrow$ (4).

Hard part of the proof was

(1) $\to$ (3) (and (2))

(3) $\to$ (4).

(4) $\to$ (1) was easy.

The point is that the proof also showed (3) $\to$ (2).

11.5.

Important corollary.

Approximation Theorem. Let $|\cdot|_1, \dots, |\cdot|_n$ be pairwise inequivalent valuations. Given $a_1, \dots, a_n \in K$ and $\varepsilon > 0$.

There exists $x \in K$ s.t.

$$|x - a_i|_i < \varepsilon \quad \text{for all } i = 1, \dots, n.$$

What does this mean?

Let $K = \mathbb{Q}$, consider $|\cdot|_3, |\cdot|_5, |\cdot|_7$, $\varepsilon = \frac{1}{10}$.

Let $a_1 = 2$, $a_2 = 3$, $a_3 = 5$.

Then there exists $x \in \mathbb{Q}$,

$$|x - 2|_3 < \frac{1}{10}$$
$$|x - 3|_5 < \frac{1}{10}$$
$$|x - 5|_7 < \frac{1}{10}$$

If $x \in \mathbb{Z}$, says same as
$$x \equiv 2 \pmod{27}$$
$$x \equiv 3 \pmod{25}$$
$$x \equiv 5 \pmod{49}.$$

(if we know $|\cdot|_3, |\cdot|_5, |\cdot|_7$ ineq.)

So it's like CRT.

But, maybe $x \in \mathbb{Q}$.

Could also throw in the real valuation.

e.g. $|x - \pi|_\infty < \frac{1}{10}$.

Here, certainly $x \in \mathbb{Z}$ not good enough!

Proof. ~~[illegible]~~.

~~[illegible]~~, ~~[illegible]~~.

Claim. There exists $z \in K$ with

$$|z|_1 > 1, \quad |z|_j < 1 \quad \text{for } j \neq 1.$$

Proof of claim for $n=2$. (two valuations)

Almost a tautology. By the extended prop.,

there are $\alpha, \beta \in K$ with

$$|\alpha|_1 < 1 \qquad |\alpha|_2 \geq 1 \qquad (\text{if } > 1 \text{ we're done})$$
$$|\beta|_2 < 1 \qquad |\beta|_1 \geq 1$$

and $\left|\frac{\alpha}{\beta}\right|_1 < 1 \qquad \left|\frac{\alpha}{\beta}\right|_2 > 1.$

Now, induct. Suppose

$$|z|_1 > 1 \qquad |z|_j < 1 \quad \text{for} \quad j = 2, \ldots, n-1.$$

If $|z|_n < 1$? done.

If $|z|_n = 1$? Take $z' = z^m y$ where m is big, $|y|_1 < 1 \quad |y|_n > 1$.

If $|z|_n > 1$? Look at $\dfrac{z^m}{1+z^m}$, converges to 1 w.r.t. $|\cdot|_1$ and $|\cdot|_n$

converges to 0 w.r.t. $|\cdot|_2, \ldots |\cdot|_{n-1}$.

Choose $z' = \dfrac{z^m}{1+z^m}\, y$, for m big.

So the sequence $\dfrac{z'^m}{1+z'^m}$ converges to 1 in $|\cdot|_1$, 0 in $|\cdot|_2, \ldots, |\cdot|_n$.

Write w_1 for this (smth very close) and similarly $w_2, \ldots, w_n$.

Then, choose $x = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$.

Then $|x - a_1|_1 = |\underbrace{a_1(w_1 - 1)}_{\text{is really small}} + a_2 w_2 + \cdots + a_n w_n|_1$

and so $< \varepsilon$ for suitable w_i.

Similar true for other valuations.