$L|K$ alg. ext of local fields.

A valuation v on K extends uniquely to w on L by

$$|\alpha| = \sqrt[n]{|N_{L/K}(\alpha)|} .$$

$e(w|v) = [w(L^\times) : v(K^\times)]$ ramification index

Have also $\mathcal{O}_L, \mathcal{O}_K$ valuation rings

$\mathfrak{m}_L, \mathfrak{m}_K$ max ideals

$\lambda = \mathcal{O}_L/\mathfrak{m}_L \quad \kappa = \mathcal{O}_K/\mathfrak{m}_K$.

$f(w|v) = [\lambda : \kappa]$ residue class degree.

Theorem. $e \cdot f = [L : K]$.

(so, $L|K$ is unramified if $[L:K] = [\lambda : \kappa]$.)

T4.5) Def$^{\text{n}}$: L/K (finite ext. of $\mathbb{Q}_p$) is <u>unramified</u>

if
$$[L:K] = [\lambda:\kappa].$$
$$\text{i.e. } e(L|K) = 1.$$

An arbitrary algebraic extension L/K is unramified if it is a union of finite unramified extensions.

Prop. (7.2) Given $L|K$, $K'|K$ inside a fixed alg closure $\overline{K}$. Then,
$$L|K \text{ unramified} \Longrightarrow L\cdot K'|K' \text{ unramified.}$$

Proof. Write $L' = L\cdot K'$

use the notation $\mathcal{O}, \mathfrak{p}, \kappa, \mathcal{O}', \mathfrak{p}', \kappa', \mathcal{O}, \mathfrak{P}, \lambda, \mathcal{O}', \mathfrak{P}', \lambda'$.

Can argue just for finite extensions.

By the primitive element theorem $\lambda = \kappa(\bar{\alpha})$ for some $\alpha \in \mathcal{O}$.

Write $f(x) \in \mathcal{O}[x]$ min poly of α. $\bar{f}(x) = f(x) \bmod \mathfrak{p} \in \kappa[x]$.

Then
$$[\lambda:\kappa] \leq \deg(\bar{f}) = \deg(f) = [K(\alpha):K] \leq [L:K] = [\lambda:\kappa]$$

so $L = K(\alpha)$ and $\bar{f}$ is the min poly of $\bar{\alpha}$ over κ.

So $L' = K'(\alpha)$.

why is $L'|K'$ unramified?

Let $g(x) \in \mathcal{O}'[x]$ min poly of α over K'.

$\bar{g}(x) = g(x) \bmod \mathfrak{p}' \in \kappa'[x]$.

Note that $\bar{g}(x)$ is a factor of $\bar{f}(x)$.

By Hensel's Lemma $\bar{g}(x)$ is irreducible.

(If it factored, would lift to a factorization of $g(x)$. ($\bar{g}(x)$ is separable.))

So $[\lambda':\kappa'] \leq [L':K'] = \deg(g) = \deg(\bar{g}) = [\kappa'(\bar{\alpha}):\kappa'] \leq [\lambda':\kappa']$.

So $[L':K'] = [\lambda':\kappa']$, $L'|K'$ unramified

(T4.6) = 15.2.

Cor.

If $L' | K$ is an unramified extension and $L \subseteq L'$, then $L | K$ is also unramified.

Proof. By prop., $L' | L$ is unramified.

Have
$$[L' : K] = [\lambda_{L'} : \kappa]$$
$$[L' : L] = [\lambda_{L'} : \lambda_L].$$

Since field degrees are multiplicative, $L|K$ is ur.
(i.e. $[L : K] = [\lambda_L : \kappa]$.)

Cor. If L and L' are unramified over K, so is LL'.

Proof. $LL' | L'$ is unramified, with
$$[\lambda_{L'} : \kappa] = [L' : K]$$
$$[\lambda_{LL'} : \lambda_{L'}] = [LL' : L'].$$

(Use : separability is transitive)

Def. Fix an algebraic closure $\bar{K}$ of K.
Then the composite of all unramified subextensions $L \subseteq \bar{K}$ of K is the maximal unramified extension T of K.

Prop. (7.5) The residue class field of T is $\bar{\kappa}$ $(= \bar{\mathbb{F}}_p)$.
Moreover, $v(T^\times) = v(K^\times)$.

Proof. See Neukirch, but this is not hard.

(Tame ramification : 7.6, 7.7, 7.8, 7.9, 7.10, 7.11)

T5.3. Def. Let $L|K$ be a finite extension of local fields.

Then, $L|K$ is tamely ramified if $\lambda|k$ is separable (automatic for exts of $\mathbb{Q}_p$), and there exists an intermediate field T with

$T|K$ unramified

$[L:T]$ coprime to p (= char k = char λ).

(Typically T = max UR sub ext of $L|K$.)

$$\begin{array}{l} L \\ \big| \;\big) \text{cop. to } p \\ T \\ \big| \;\big) \text{ur} \\ K \end{array}$$

Note: UR extensions are tamely ramified.

Prop. 7.7. $L|K$ is tamely ramified iff $L|T$ is generated by radicals

$$L = T(\sqrt[m_1]{a_1}, \dots, \sqrt[m_r]{a_r}) \quad \text{with } (m_i, p) = 1.$$

"Tame" = not too bad.

Go to the Jones - Roberts database.

Look at deg n exts of $\mathbb{Q}_p$ when $p \nmid n$. Then $p | n$.

especially for n also prime.

Very compelling..

Extensions of valuations. (N. 2.8.)

Bring back the global fields.

A local field has one valuation

A global field has a lot.

Interested in completions also.

(i.e. $\mathbb{Q}$ has the p-adic valuation

$\mathbb{Q}_p$ the completion).

TS.4. Given K, number field.

valuation v, completion K_v, alg. closure $\overline{K_v}$.

Recall v extends uniquely and canonically to K_v (call it v again)

and to $\overline{K_v}$. (call it $\bar{v}$)

Now, given L/K, $_\wedge$ valuation v on L. we have an embedding

$$L \overset{\tau}{\hookrightarrow} \overline{K_V} \quad \text{fixing } K.$$

(say a little bit..)

Restrict the valuation $\bar{v}$ to $\tau(L)$.

Label this valuation w. $(w = \bar{v} \circ \tau.)$

Can write this as $|x|_w = |\tau(x)|_{\bar{v}}$ for $x \in L$.

This map is continuous, and it extends uniquely to a continuous K-embedding

$$L_w \overset{\tau}{\hookrightarrow} \overline{K_V}$$

$$x = w\text{-}\lim_{n\to\infty} x_n \longrightarrow \tau x := \bar{v}\text{-}\lim_{n\to\infty} \tau(x_n).$$

(Cauchy sequences map to Cauchy sequences.)

Have a field diagram

$$\begin{tikzcd} & L_w \arrow[d, dash] \\ L \arrow[ur, dash] \arrow[d, dash] & K_V \\ K \arrow[ur, dash] & \end{tikzcd}$$

Canonical
$_\wedge$ Extension of w from L to L_w is the unique extension of v from K_v to L_w.

The extension L_w satisfies $L_w = LK_v$.

Why is this? $LK_v \subseteq L_w$ is again complete (4.8 - part we skipped proving)

contains L and so must be its completion.

As we saw before, $|x|_w = \sqrt[n]{|N_{L_w|K_v}(x)|_v}$.

This represents a <u>local</u> - to - <u>global</u> principle.

(Motivating example: $K = \mathbb{C}(t)$

L = alg. fns on some Riemann surface

Pass to K_v and L_w: look at power series

<u>local</u> study of functions.

(<u>Take Jesse Kass's course</u>)

Now the embedding $L \xhookrightarrow{\tau} \overline{K_v}$ was not necessarily unique.

There might be other such embeddings.

And, we got w from τ.

<u>Example</u>. Let $L/K = \mathbb{Q}(i)/\mathbb{Q}$.

There are two embeddings $\mathbb{Q}(i) \hookrightarrow \mathbb{Q}_5$.

How to find them? $2^2 \equiv -1 \mod 5$.

$3^2 \equiv -1 \mod 5$.

By Hensel's lemma, they lift uniquely.

Choose either for image of i.

There is no embedding $\mathbb{Q}(i) \hookrightarrow \mathbb{Q}_7$.

But there are two embeddings $\mathbb{Q}(i) \hookrightarrow \overline{\mathbb{Q}_7}$.

Once we <u>fix</u> an algebraic closure of $\mathbb{Q}_7$ they are distinguished

The image is $\mathbb{Q}_7[x]/(x^2+1)$ again write $\mathbb{Q}_7(i)$.

Corollary. We have

(1) $$[L:K] = \sum_{w|v} [L_w : K_v]$$

(2) $$N_{L/K}(\alpha) = \prod_{w|v} N_{L_w/K_v}(\alpha) \qquad Tr_{L/K}(\alpha) = \sum_{w|v} Tr_{L_w/K_v}(\alpha).$$

(1) is immediate.

(2): On $L \otimes_K K_v \simeq \prod_{w|v} L_w$

look ~~at~~ the endomorphism multiplication by α.

Char poly of α is the same on:

K-vector space L

K_v-vector space $L \otimes K_v$.

So
$$\text{char poly}_{L/K}(\alpha) = \prod_{w|v} \text{char poly}_{L_w/K_v}(\alpha)$$
and we get (2).

efg for valuations.

Recall, for $w|v$, $e(w|v) = e_w = (w(L^\times) : v(K^\times))$

$f(w|v) = f_w = [\lambda_w : \kappa_v]$,

and $[L_w : K_v] = e_w \cdot f_w$.

(Prop 6.8: "ef" for local fields)

Therefore:

Theorem 8.5. $$\sum_{w|v} e(w|v) f(w|v) = [L:K].$$

T6.6. This is what we saw before.

Given L/K, $\mathfrak{p}$ in K, with $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$,

$\mathfrak{p} \longleftrightarrow$ the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$ of K.

($v_\mathfrak{p}(a)$ = # of $\mathfrak{p}$'s in the ideal factorization of a)

$\mathfrak{P}_i \longleftrightarrow$ the valuations $w_{\mathfrak{P}_i}$ extending $v_\mathfrak{p}$.

Check: the $\left\langle \begin{array}{l} \text{inertia degrees} \\ \text{ramification indices} \end{array} \right\rangle$ match up

And so says the same thing as $\sum_{i=1}^{r} e_i f_i = [L:K]$.

T6.4. **Theorem.** ~~We know~~ φ gives an isomorphism

$$L \otimes_K K_v \xrightarrow{\sim} \prod_{w|v} L_w .$$

Proof. Let $L = K(\alpha)$, and write (as before) $f(X) = \prod_{w|v} f_w(X)$
$f(X)$ min poly of α over $K_v(X)$.

Now, consider all the L_w embedded in $\overline{K_v}$,
write α_w: image of α under $L \to L_w$, so that
$L_w = K_v(\alpha_w)$, and
$f_w(X)$ is the min. polynomial of α_w over K_v.

Commutative diagram:

$$\begin{tikzcd}
K_v[X]/(f) \arrow[r] \arrow[d] & \prod_{w|v} K_v[X]/(f_w) \arrow[d] \\
L \otimes_K K_v \arrow[r] & \prod_{w|v} L_w
\end{tikzcd}$$

Top is an iso by CRT.
Right: $X \to \alpha_w$, an iso. because $K_v[X]/(f_w) \cong K_v(\alpha_w) = L_w$.

Left: $X \to \alpha \otimes 1$, iso. because $K_v[X]/(f) \cong K(\alpha) = L$.
(extension of scalars!)

Everything commutes, so bottom is an iso also.

T6.3. Valuations and polynomials.

Given $L = K(\alpha)$ where α is a zero of $f(x) \in K[X]$.

Write, in K_v, $f(x) = f_1(x)^{m_1} \cdots f_r(x)^{m_r}$. ($m_i$ are 1 in the separable case)

How to get a K-embedding $\tau : L \to \overline{K_v}$?

$$\tau : L \longrightarrow \overline{K_v}$$
$$\alpha \longrightarrow \beta, \text{ where } \beta \text{ is a zero of } f(X) \text{ in } \overline{K_v}.$$

Two embeddings τ, τ' are conjugate iff the β's chosen are roots of the same irreducible $f_i(x)$.

Theorem 8.2. With the above, the valuations $w_1, \dots, w_r$ extending v to L are in bijection with the f_i above.

Moreover, we see how to get them:

Take $\alpha_i \in \overline{K_v}$ a zero of some f_i.

$$\tau_i : L \longrightarrow \overline{K_v} \quad \text{a } K\text{-embedding.}$$
$$\alpha \longrightarrow \alpha_i$$

Then $w_i = \bar{v} \circ \tau_i$, and

τ_i extends to an isomorphism $\tau_i : L_{w_i} \xrightarrow{\sim} K_v(\alpha_i)$.

Valuations and tensor products.

With above, get a hom. $L \otimes_K K_v \xrightarrow{\varphi_w} L_w$
$$a \otimes b \longrightarrow ab.$$

What "is" $L \otimes_K K_v$? First of all, it's a K_v-vector space.
But it's a K_v-algebra, multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

Do this for all w, obtain $L \otimes_K K_v \xrightarrow{\varphi} \prod_{w|v} L_w$.

T.6.2.

Given τ and τ', with $\tau' = \sigma \circ \tau$ for $\sigma \in Gal(\overline{K_v} | K_v)$.
(two embeddings conjugate over K_v)

Well, $\bar{v}$ is the <u>only</u> extension of v from K_v to $\overline{K_v}$

So $\bar{v} = \bar{v} \circ \sigma$. So $\bar{v} \circ \tau = \bar{v} \circ (\sigma \circ \tau)$.

Conversely: Given $\tau, \tau' : L \hookrightarrow \overline{K_v}$ K-embeddings, s.t.
$\bar{v} \circ \tau = \bar{v} \circ \tau'$.

Define a K-isomorphism $\sigma : \tau L \longrightarrow \tau' L$
$\sigma = \tau' \circ \tau^{-1}$.

[Then σ extends to a K_v-isomorphism
$\sigma : \tau L \cdot K_v \longrightarrow \tau' L \cdot K_v$.

Why? τL is dense in $\tau L \cdot K_v$, so every $x \in \tau L \cdot K_v$ is a limit $x = \lim_{n \to \infty} \tau x_n$ with $x_n \in L$.

with $\tau' x_n = \sigma \tau x_n$, because of $\bar{v} \circ \tau = \bar{v} \circ \tau'$ we have

$$\sigma x := \lim_{n \to \infty} \sigma \tau x_n .$$

In other words, equality of valuations guarantees we get a Cauchy sequence. <u>Define</u> σx to be the RHS.

Easily checked, the map $x \to \sigma x$ does not depend on the x_n chosen, so we get an isomorphism

$$\tau L \cdot K_v \xrightarrow{\sigma} \tau' L \cdot K_v$$ leaving K_v fixed.

This is our Galois element:
Extend σ (arbitrarily) to a K_v-automorphism $\tilde{\sigma} \in G(\overline{K_v} | K_v)$

Get $\tau' = \tilde{\sigma} \circ \tau$

so τ, τ' are conjugate over K_v.

T.6.1. Local and global fields.

Interested in the following.

L/K extension of global fields (here: number fields) ($[L:K]$ finite)
(see Neukirch for gen. case)

v: valuation on K.

w: valuation on L extending v.

How to get this?

Choose an embedding $L \overset{\tau}{\hookrightarrow} \overline{K_v}$

Get a valuation $\bar{v}$ on K_v.

Use this embedding to obtain w on L.

Above extends to a continuous map $L_w \hookrightarrow \overline{K_v}$.

$$\begin{tikzcd} & L_w \arrow[d, no head] \\ L \arrow[ur, no head] \arrow[d, no head] & K_v \\ K \arrow[ur, no head] & \end{tikzcd}$$

$L_w = LK_v$, and $|x|_w = \sqrt[n]{|N_{L_w|K_v}(x)|}$.

Theorem. (Extension Theorem 8.1)

Given the above,

(1) Every extension w of v arises as the composite $w = \bar{v} \circ \tau$ for some K-embedding $\tau: L \hookrightarrow \overline{K_v}$.

(2) Two extensions $\bar{v} \circ \tau$, $\bar{v} \circ \tau'$ are equal if and only if τ and τ' are conjugate over K_v.

Proof. (1) Choose some $w|v$, form L_w.

Choose some K_v-embedding $\tau: L_w \hookrightarrow \overline{K_v}$, then by construction $\bar{v} \circ \tau$ must coincide with w.

Restricting to L gives what we want.