T1.5. = T2.2.

Important corollary.

Approximation Theorem. Let $|\cdot|_1, \dots, |\cdot|_n$ be pairwise inequivalent valuations, Given $a_1, \dots, a_n \in K$ and $\varepsilon > 0$.

There exists $x \in K$ s.t.

$$|x - a_i|_i < \varepsilon \quad \text{for all} \quad i = 1, \dots, n.$$

What does this mean?

Let $K = \mathbb{Q}$, consider $|\cdot|_3, |\cdot|_5, |\cdot|_7$, $\varepsilon = \frac{1}{10}$.

Let $a_1 = 2$, $a_2 = 3$, $a_3 = 5$.

Then there exists $x \in \mathbb{Q}$,

$$|x - 2|_3 < \frac{1}{10}$$
$$|x - 3|_5 < \frac{1}{10}$$
$$|x - 5|_7 < \frac{1}{10}$$

If $x \in \mathbb{Z}$, says same as
$$x \equiv 2 \pmod{27}$$
$$x \equiv 3 \pmod{25}$$
$$x \equiv 5 \pmod{49}.$$

(if we know $|\cdot|_3, |\cdot|_5, |\cdot|_7$ ineq.)

So it's like CRT.

But, maybe $x \in \mathbb{Q}$.

Could also throw in the real valuation.

e.g. $|x - \pi|_\infty < \frac{1}{10}$.

Here, certainly $x \in \mathbb{Z}$ not good enough!

Proof. ~~[illegible]~~,

~~[illegible]~~

Claim. There exists $z \in K$ with

$$|z|_1 > 1, \quad |z|_j < 1 \quad \text{for } j \neq 1.$$

T1.6. = T2.3.

Proof of claim for $n=2$. (two valuations)

Almost a tautology. By the extended prop.,
there are $\alpha, \beta \in K$ with

$$|\alpha|_1 < 1 \qquad |\alpha|_2 \geq 1 \qquad (\text{if } > 1 \text{ we're done})$$

$$|\beta|_2 < 1 \qquad |\beta|_1 \geq 1$$

and $\left|\frac{\alpha}{\beta}\right|_1 < 1 \qquad \left|\frac{\alpha}{\beta}\right|_2 > 1.$

Now, induct. Suppose

$$|z|_1 > 1 \qquad |z|_j < 1 \quad \text{for} \quad j = 2, \ldots, n-1.$$

If $|z|_n < 1$? done.

If $|z|_n = 1$? Take $z' = z^m y$ where m is big,
$|y|_1 < 1 \quad |y|_n > 1$.

If $|z|_n > 1$? Look at $\dfrac{z^m}{1+z^m}$, converges to 1 w.r.t. $|\cdot|_1$ and $|\cdot|_n$
converges to 0 w.r.t. $|\cdot|_2, \ldots |\cdot|_{n-1}$

Choose $z' = \dfrac{z^m}{1+z^m} y$, for m big.

So the sequence $\dfrac{z'^m}{1+z'^m}$ converges to 1 in $|\cdot|_1$
0 in $|\cdot|_2, \ldots, |\cdot|_n$

Write w_1 for this (smth very close) and similarly $w_2, \ldots, w_n$.

Then, choose $x = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$.

Then $|x - a_1|_1 = |a_1 \underbrace{(w_1 - 1)}_{\text{is really small}} + a_2 w_2 + \cdots + a_n w_n|_1$

and so $< \varepsilon$ for suitable w_i

Similarly ...

Prop. (3.7) Every valuation of $\mathbb{Q}$ is equivalent to one of the valuations $|\cdot|_p$ or $|\cdot|_\infty$

Some general setup and results. (Same proofs as for $\mathbb{Z}_p$, $\mathbb{Q}_p$.)
(See N., Ch. 3 - 4.)

Proposition. Let K be a field with valuation $v(-)$
and absolute value $|\cdot| = q^{-v(-)}$
for some $q>1$.
(Recall: different choices of q: equiv. valuations)

The subset
$$\mathcal{O} = \{x \in K : v(x) \geq 0\} = \{x \in K : |x| \leq 1\}$$
(the valuation ring of K w.r.t. $|\cdot|$)
is a ring with group of units
$$\mathcal{O}^\times = \{x \in K : v(x) = 0\} = \{x \in K : |x| = 1\}$$
and unique maximal ideal
$$\mathfrak{p} = \{x \in K : v(x) \geq 1\} = \{x \in K : |x| < 1\}.$$

The valuation is discrete if it has a smallest positive value s, and normalized if $s = 1$.
Dividing by s, can always pass to a normalized valuation.

The prime elements are those in $\mathfrak{p} \setminus \mathfrak{p}^2$.
Writing π for an arbitrary prime elt.,
every $x \in K^\times$ can be written uniquely as $x = u \cdot \pi^m$
for $u \in \mathcal{O}^\times$, $m \in \mathbb{Z}$.

(If $v(x) = m$, then $v(x\pi^{-m}) = 0$ so it is a unit.)

T2.5.

If v is a discrete valuation, then $\mathcal{O}$ is a PID with a unique maximal ideal; i.e. a discrete valuation ring.

The ideals are $\mathfrak{p}^n$, for $n \in \mathbb{Z}$, and we have

$$K^\times = (\pi) \times \mathcal{O}^\times.$$

(In fact, $K^\times = (\pi) \times \underbrace{\mu_{q-1}}_{\text{roots of unity}} \times \underbrace{U^{(1)}}_{\substack{\text{principal units}\\ 1+\mathfrak{p}}}$)

We let $\hat{\mathcal{O}}$ be the completion of $\mathcal{O}$ w.r.t. $|\cdot|$.

Then the maximal ideal of $\hat{\mathcal{O}}$ is $\hat{\mathfrak{p}}$, and

$$\hat{\mathcal{O}} / \hat{\mathfrak{p}}^n \cong \mathcal{O}/\mathfrak{p}^n \quad \text{for every } n \geq 1.$$

Moreover, we have an isomorphism and homeomorphism

$$\hat{\mathcal{O}} \longrightarrow \varprojlim_n \mathcal{O}/\mathfrak{p}^n.$$

So, the question is:

Given $K/\mathbb{Q}$, can cook up valuations on K. Complete with respect to them. Get "local fields".

Can take the opposite approach. Start with $\mathbb{Q}_p$. Consider an algebraic extension. Do we get the same?

e.g. $\sqrt{-1} \notin \mathbb{Q}_7$.

$$\begin{tikzcd} & \mathbb{Q}_7(i) \arrow[d, no head] \\ \mathbb{Q} \arrow[r, squiggly] & \mathbb{Q}_7 \end{tikzcd}$$

Is it complete?

Indeed, is it the completion of $\mathbb{Q}(i)$ at (7)? Yup!

Goal: understand extensions of valuations.

T3.1. Extensions of local fields.

Def. A field K is local if:
- it is complete w.r.t. a discrete valuation;
- it has a finite residue class field.

Theorem. (N 2.5.2) These are precisely the finite extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$.

Theorem. They all satisfy Hensel's Lemma.
(see N 2.6, "Henselian fields").

Applications of Hensel.

Prop. We have $\mathbb{Q}_p \supseteq \mu_{p-1}$ ($p-1$ th roots of unity).

Proof. $(\mathbb{Z}/p)^\times$ is an abelian group of size $p-1$.
That means $a^{p-1} \equiv 1 \mod p$ for all $a \in \mathbb{Z}_p^\times$.
So $x^{p-1} - 1 \in \mathbb{Z}_p[x]$ splits completely in $\mathbb{F}_p[x]$.
into distinct factors
By Hensel, it splits into distinct factors in $\mathbb{Z}_p[x]$ too.

Prop. (4.7) Let K be complete w.r.t. nonarch. $|\cdot|$.
(e.g. $K = \mathbb{Q}_p$)
For every irred. polynomial $f(x) = a_0 + a_1 x + \dots + a_n x^n \in K[x]$ with $a_0 a_n \neq 0$, one has

$$|f| = \max\{|a_0|, |a_n|\}.$$

(Here $|f| = \max_i |a_i|$.)

lower case
In particular, writing $\mathcal{O}$ for the valuation ring of K, $a_n = 1$ and $a_0 \in \mathcal{O}$ imply $f \in \mathcal{O}[x]$.

T3.2.

Proof. By multiplying through by an element of K, can assume $f \in \mathcal{O}[x]$ and $|f| = 1$.

In the list $a_0, a_1, a_2, \ldots$ let a_r be the first which has $|a_r| = 1$.

Then, mod $\mathfrak{p}$ ($\mathfrak{p}$ = max ideal of $\mathcal{O}$),

$$f(x) \equiv a_r x^r + a_{r+1} x^{r+1} + \cdots + a_n x^n \mod \mathfrak{p}$$

$$= x^r (a_r + a_{r+1} x + \cdots + a_n x^{n-r})$$

If $\max\{|a_0|, |a_n|\} < 1$, this is a nontrivial factorization into coprime polynomials.

By Hensel it lifts from $\mathcal{O}/\mathfrak{p}$ to $\mathcal{O}$. Contradiction.

Big Theorem. (4.8) Let K be complete w.r.t. $|\cdot|$. Let L/K be any algebraic extension. Then $|\cdot|$ extends uniquely to L, with

$$|\alpha| = \sqrt[n]{|N_{L/K}(\alpha)|}. \qquad (n := [L:K] \text{ when } L/K \text{ finite})$$

Then L is also complete.

Proof. Assume:

- $|\cdot|$ is nonarchimedean (otherwise K is $\mathbb{R}$ or $\mathbb{C}$)
- L/K is finite. (Can assume WLOG: Prove for $K(\alpha)$, take compositum over all $\alpha \in L$.)

T3.3.

Notation.

$$\begin{array}{lllll} L & \supseteq & \mathcal{O} & \left(\begin{array}{l}\text{int closure of } \mathfrak{o} \text{ in } L. \\ \text{will prove: is valuation ring of } L.\end{array}\right) & \supseteq \mathcal{P} \text{ (later)} \\ | & & & & \\ K & \supseteq & \mathfrak{o} & \text{(valuation ring)} & \supseteq \mathfrak{p} \text{ (unique max ideal)} \end{array}$$

Note. $\mathcal{O}$ and $\mathfrak{o}$ are easy to confuse. Sorry.

Proof. (existence:)

Let $\mathcal{O}$ be int. closure of $\mathfrak{o}$ in L.

Claim. $\mathcal{O} = \{\alpha \in L : N_{L/K}(\alpha) \in \mathfrak{o}\}$.

Proof of claim.

$\subseteq$: Given $\alpha \in \mathcal{O}$, α^m satisfies a monic poly in $\mathfrak{o}$, norm is $\pm$ (its last coefficient) for some m.

$\supseteq$: Given $\alpha \in L^*$ with $N_{L/K}(\alpha) \in \mathfrak{o}$.

Let $f(x) = x^d + a_{d-1}x^{d-1} + \dots + a_0 \in K[x]$ min poly of α.

Then $N_{L/K}(\alpha) = \pm a_0^m$, so $|a_0| \le 1$ (i.e. $a_0 \in \mathfrak{o}$).

USE PROPOSITION 4.7 : $f(x) \in \mathfrak{o}[x]$

By def., $\alpha \in \mathcal{O}$.

Now define $|\alpha| := \sqrt[n]{|N_{L/K}(\alpha)|}$. (Note: if $\beta \in K$, $\sqrt[n]{|N_{L/K}(\beta)|} = \sqrt[n]{|\beta|^n} = |\beta|$)

Easy: $|\alpha| = 0 \longleftrightarrow \alpha = 0$

$|\alpha\beta| = |\alpha||\beta|$.

Want to check strong triangle inequality

$$|\alpha + \beta| \le \max\{|\alpha|, |\beta|\}.$$

Assume WLOG $|\beta| \le |\alpha|$, divide by $|\beta|$, enough to check

$$|\alpha + 1| \le \max\{|\alpha|, 1\}$$

i.e. $|\alpha + 1| \le 1$ if $|\alpha| \le 1$.

T3.4.

By claim, this reduces to $\alpha \in \mathcal{O} \implies \alpha + 1 \in \mathcal{O}$.
But this is trivial. (integral elts are a ring)

Therefore, $|\alpha| = \sqrt[n]{|N_{L/K}(\alpha)|}$ defines a valuation on L which agrees with old valuation on K.

Moreover, $\mathcal{O}$ is the valuation ring by our claim

Uniqueness. Suppose $|\cdot|'$ is another ext w/ valuation ring $\mathcal{O}'$.
Let $\underline{P}, \underline{P}'$: max ideals of $\mathcal{O}, \mathcal{O}'$.

Claim. $\mathcal{O} \subseteq \mathcal{O}'$.

Proof. Note $\mathcal{O}, \mathcal{O}'$ are both in L (by construction).
Given $\alpha \in \mathcal{O} \setminus \mathcal{O}'$ with min poly

$$f(x) = x^d + a_{d-1} x^{d-1} + \cdots + a_0 .$$

Then all the a's are in $\mathcal{O}$, and $\alpha^{-1} \in \underline{P}'$.
(because it is not in $\mathcal{O}'$)

Plug in α.

$$0 = \alpha^d + a_{d-1}\alpha^{d-1} + \cdots + a_0$$

$$0 = 1 + \underbrace{a_{d-1}\alpha^{-1} + \cdots + a_0 \alpha^{-d}}_{\text{This is in } \underline{P}',\text{ so } 1 \text{ is also, contradiction.}}$$

Thus, $\mathcal{O} \subseteq \mathcal{O}'$, i.e. $|\alpha| \le 1 \implies |\alpha|' \le 1$.
By the approximation theorem, $|\cdot|$ and $|\cdot|'$ are equivalent.
i.e. $|\cdot| = (|\cdot|')^s$ for some $s > 0$.
Since they agree on K, they are equal.

L is complete with respect to this valuation:
Proof omitted; see N. 2.4.9.

So. Extend valuations from K to L. $[L:K]=n$

For absolute values, $|\alpha| = \sqrt[n]{|N_{L/K}(\alpha)|}$.

In terms of (additive valuations),
a valuation v on K extends to a valuation w on L satisfying

$$w(\alpha) = \frac{1}{n} v(N_{L/K}(\alpha)).$$

Note also, if v is normalized s.t. $v(K^*) = \mathbb{Z}$,

then $\frac{1}{n}\mathbb{Z} \subseteq w(L^\times) \subseteq \mathbb{Z}$.

Example. Let $p = 7$, $K = \mathbb{Q}_p$, $L = \mathbb{Q}_p(\sqrt{p})$.

Then for $\alpha \in L$, $|\alpha| = \sqrt{|N_{L/\mathbb{Q}_p}(\alpha)|}$.

In particular, $|\sqrt{p}| = \sqrt{|N(\sqrt{p})|}$

$$= \sqrt{|\sqrt{p}\cdot(-\sqrt{p})|}$$

$$= \sqrt{|-p|} = \sqrt{p}.$$

The same calculation gives $w(\sqrt{p}) = \frac{1}{2}$,
where w is the extended valuation.

Example. Let $p=7$, $K = \mathbb{Q}_p$, $L = \mathbb{Q}_p(\sqrt{3})$.
(Check: 3 is not a quad. residue.)

Then $N_{L/K}(a+b\sqrt{3}) = a^2 - 3b^2$.
Check: If this is divisible by 7, it is divisible by 7^2.

Thus. $w(L) = \mathbb{Z}$.

T3.6. Definition.

The index $[w(L^\times) : v(K^\times)]$ is called the ramification index of L/K.

Write $e(w|v)$.

Def. Given L/K w/ valuation rings $\mathcal{O} | o$.

max ideals $\mathfrak{P} | \mathfrak{p}$.

Have residue fields $\lambda := \mathcal{O}/\mathfrak{P}$

$\kappa := o/\mathfrak{p}$.

As before $\kappa \hookrightarrow \lambda$ and λ is a finite ext.

The degree $[\lambda : \kappa]$ is the inertia degree of L/K.

Write it $f(w|v)$.

Theorem. If L/K is finite separable, v is a disc. valuation on K, w extends it, then

$$[L : K] = e(w|v) \cdot f(w|v).$$

(Ponder: where did the g go?)

T4.1.

Last time.

Suppose K is complete w.r.t. $|\cdot|$.

L/K alg extension.

Then $|\cdot|$ may be uniquely extended to L, with

$$|\alpha| = \sqrt[n]{|N_{L/K}(\alpha)|}.$$

L is again complete w.r.t. $|\cdot|$.

In terms of additive valuations,

get a valuation w prolonging the valuation v on K, with

$$w(\alpha) = \frac{1}{n} v(N_{L/K}(\alpha)).$$

So,

$$\frac{1}{n} v(K^\times) \supseteq w(L^\times) \supseteq v(K^\times).$$

Def. $e(w|v) := [w(L^\times) : v(K^\times)]$ is the ramification index of L/K (of $w|v$).

Let $\mathcal{O}$ and o be the valuation rings, $\lambda := \mathcal{O}/\underline{m}_L$ and $\kappa := o/\underline{m}_K$ the residue class fields.

We have an injection $\kappa \hookrightarrow \lambda$:

$$o/\underline{m}_K \to \mathcal{O}/\underline{m}_L$$

$$x \to x.$$

Well defined because $\underline{m}_K \cdot \mathcal{O} \subseteq \underline{m}_L$.

Injective because 1 is not in the kernel.

Def. $f(w|v) := [\lambda : \kappa]$ is the residue class degree.

T4.2. Remark.

Let Π and π be prime elements of $\mathcal{O}$ and $\mathfrak{o}$.

Then $w(L^\times) = w(\Pi)\cdot\mathbb{Z}$, $w(K^\times) = \cancel{w}(\pi)\cdot\mathbb{Z}$.

$$e = [w(\Pi)\mathbb{Z} : \cancel{w}\, v(\pi)\mathbb{Z}],$$

so that $v(\pi) = e\cdot w(\Pi)$, i.e.

$$\pi = \varepsilon\cdot\Pi^e \quad \text{for some } \varepsilon\in\mathcal{O}^\times.$$

In particular we see that $\mathfrak{p}\mathcal{O} = \pi\mathcal{O} = \Pi^e\mathcal{O} = \underline{P}^e$,

i.e. $\mathfrak{p} = \underline{P}^e$.

Theorem. (6.8) Assume L/K is finite separable and $|\cdot|$ discrete.

Then $[L:K] = ef$.

Proof. (1) Show $ef \le [L:K]$.

Let $w_1,\dots,w_f$ be representatives of a basis for $\lambda|\kappa$. (i.e. they live in $L^\times$)

$1,\dots,\Pi^{e-1}$ ~~a basis~~ $\in L^\times$ representing ~~a basis~~ all the cosets of $[w(L^\times) : w(K^\times)]$.

Want to show the $w_j\cdot\Pi^i$ are (1) linearly independent/K
(2) a basis of L/K.

To show (1), write

$$\sum_{i=0}^{e-1}\sum_{j=1}^{f} a_{ij}w_j\Pi^i = 0, \quad a_{ij}\in K.$$

If not all a_{ij} are 0, then some $s_i := \sum_{j=1}^{f} a_{ij}w_j$ is not zero.

(because the Π^i are certainly linearly independent over K.)

T4.3. Claim. If $s_i \neq 0$ then $w(s_i) \in v(K^\times)$.

Proof. Given $\sum_{j=1}^{f} a_{ij} w_j \neq 0$,

divide by the a_{iv} of minimum value.

Get $s_i' = \frac{s_i}{a_{iv}} = \sum_i \underbrace{a_{ij}'}_{\text{These are in } \mathcal{O} \subseteq K.} \underbrace{w_j}_{\text{These are in } L}$

The w_j represent a basis for λ / k.

Therefore, s_i' can only be $0 \pmod{\mathfrak{P}}$ if all a_{ij}' are $0 \pmod{\mathfrak{p}}$.

But we divided by a_{iv} of min value, so $a_{iv}' = 1$,

so $s_i' \not\equiv 0 \pmod{\mathfrak{P}}$ and so is a unit in $\mathcal{O}$.

This implies $w(s_i) = w(a_{iv}) \in v(K^\times)$ (because $a_{iv} \in K$).

[Note: We're really using everything!]

Now, we had a sum $0 = \sum_{i=0}^{e-1} s_i \pi^i$.

Two summands must have the same valuation,

because $w(x) \neq w(y) \Rightarrow w(x+y) = \min\{w(x), w(y)\}$.

However, the s_i all have valuations in $v(K^\times)$

the π^i all represent distinct cosets of $v(K^\times)$ in $w(L^\times)$

This is a contradiction.

Proves linear independence, i.e. $ef \leq [L:K]$.

T4.4. (2). Need:

Nakayama's Lemma. Let A be a local ring with maximal ideal $\mathfrak{m}$.

Let M be an A-module, $N \subseteq M$ a submodule with M/N finitely generated.

Then, $M = N + \mathfrak{m} M \implies M = N$.

(Proof. Exercise)

To do (2), consider the $\mathfrak{o}$-module

$$M := \sum_{i=0}^{e-1} \sum_{j=1}^{f} \mathfrak{o}\, w_j \pi^i.$$

Will argue that $M = \mathcal{O}$, i.e., $\{w_j \pi^i\}$ are not only linearly dependent, but an integral basis for $\mathcal{O}|\mathfrak{o}$.

Write $$N = \sum_{j=1}^{f} \mathfrak{o}\, w_j,$$

$$M = N + \pi N + \pi^2 N + \dots + \pi^{e-1} N.$$

Then we have $\mathcal{O} = N + \pi \mathcal{O}$.

Why? For $a \in \mathcal{O}$, look at $a \bmod \pi\mathcal{O}$.

Get $a_1 w_1 + \dots + a_f w_f \pmod{\pi\mathcal{O}}$ for some $a_i \in \mathfrak{o}$.

Residue can be represented by smth of valuation 0, and all such elts. are spanned by a basis of $\lambda : \kappa$.

(In other words: $w_1, \dots, w_f$ are a basis for $\mathcal{O}/\mathfrak{P}$ over $\mathfrak{o}/\mathfrak{p}$.)

(So: a_i are only determined up to $\mathfrak{p}$.)

So, $\mathcal{O} = N + \pi\mathcal{O} = N + \pi(N + \pi\mathcal{O}) = \dots$

$= N + \pi N + \dots + \pi^{e-1} N + \pi^e \mathcal{O}$

i.e. $\mathcal{O} = M + \mathfrak{P}^e = M + \mathfrak{p}\mathcal{O}$.

Now $\mathcal{O}|\mathfrak{o}$ is finitely generated (has an integral basis) so Nakayama applies and $\underline{\mathcal{O} = M}$.

T4.5. Def. L/K (finite ext. of $\mathbb{Q}_p$) is <u>unramified</u>

if $$[L:K] = [\lambda:\kappa].$$

i.e. $e(L|K) = 1$.

An arbitrary algebraic extension L/K is unramified if it is a union of finite unramified extensions.

Prop. (7.2) Given $L|K$, $K'|K$ inside a fixed alg closure $\overline{K}$. Then,

$$L|K \text{ unramified} \implies L\cdot K'|K \text{ unramified}.$$

Proof. Write $L' = L\cdot K'$

use the notation $\mathfrak{o}, \mathfrak{p}, \kappa, \mathfrak{o}', \mathfrak{p}', \kappa', \mathcal{O}, \mathfrak{P}, \lambda, \mathcal{O}', \mathfrak{P}', \lambda'$.

Can argue just for finite extensions.

By the primitive element theorem $\lambda = \kappa(\bar{\alpha})$ for some $\alpha \in \mathcal{O}$.

Write $f(x) \in \mathfrak{o}[x]$ min poly of α. $\bar{f}(x) = f(x) \bmod \mathfrak{p}$
$\in \kappa[x]$.

Then

$$[\lambda:\kappa] \leq \deg(\bar{f}) = \deg(f) = [K(\alpha):K] \leq [L:K] = [\lambda:\kappa]$$

so $L = K(\alpha)$ and $\bar{f}$ is the min poly of $\bar{\alpha}$ over κ.

So $L' = K'(\alpha)$.

why is $L'|K'$ unramified?

Let $g(x) \in \mathfrak{o}'[x]$ min poly of α over K'.

$\bar{g}(x) = g(x) \bmod \mathfrak{p}' \in \kappa'[x]$.

Note that $\bar{g}(x)$ is a factor of $\bar{f}(x)$.

By Hensel's Lemma $\bar{g}(x)$ is irreducible.

(If it factored, would lift to a factorization of $g(x)$.

($\bar{g}(x)$ is separable.)

So $[\lambda':\kappa'] \leq [L':K] = \deg(g) = \deg(\bar{g}) = [\kappa'(\bar{\alpha}):\kappa'] \leq [\lambda':\kappa']$.

So $[L':K'] = [\lambda':\kappa']$, $L'|K'$ <u>unramified</u>

T4.6. Cor.

If $L'|K$ is an unramified extension and $L \subseteq L'$, then $L|K$ is also unramified.

Proof. By prop., $L'|L$ is unramified.

Have
$$[L':K] = [\lambda_{L'}:\kappa]$$
$$[L':L] = [\lambda_{L'}:\lambda_L].$$

Since field degrees are multiplicative, $L|K$ is ur.
(i.e. $[\cancel{L}:K] = [\lambda_L : \kappa]$.)

Cor. If L and L' are unramified over K, so is LL'.

Proof. $LL'|L'$ is unramified, with
$$[\lambda_{L'}:\kappa] = [L':K]$$
$$[\lambda_{LL'}:\lambda_{L'}] = [LL':L'].$$

(Use: separability is transitive)

Def. Fix an algebraic closure $\overline{K}$ of K.
Then the composite of all unramified subextensions $L \subseteq \overline{K}$ of K is the maximal unramified extension T of K.

Prop. (7.5) The residue class field of T is $\overline{\kappa}$ ($= \overline{\mathbb{F}_p}$).
Moreover, $v(T^\times) = v(K^\times)$.

Proof. See Neukirch, but this is not hard.

(Tame ramification: 7.6, 7.7, 7.8, 7.9, 7.10, 7.11)