Math 554/703I - Analysis I
Test 1 - **Solutions**

1. List the axioms for the real numbers.

   [See course lecture notes.]

2. Let A be an nonempty subset of $I\!R$.

   a.) Define 'upper bound' for A.

   b.) Define 'least upper bound' for A.

   c.) Prove that least upper bounds are unique.

   [Suppose that α, β are both least upper bounds for A. Since α is an upper bound for A, and β is a least upper bound of A, then $\beta \le \alpha$. By symmetry, $\alpha \le \beta$]

3. Prove for each $a \in I\!R$, $(-a) = (-1) \cdot a$.

   [See course lecture notes: But use the fact that additive inverses are unique and observe that $a + ((-1)(a)) = a(1 + (-1)) = a \cdot 0 = 0$.]

4. Suppose that $I\!R$ is an ordered field, prove

   a) Multiplicative inverses are unique.

   [If α, β are both a multiplicative inverse of an element a, then $\alpha = \alpha 1 = \alpha(a\beta) = (\alpha a)\beta = \beta$.]

   b) $(ab)^{-1} = a^{-1}b^{-1}$

   [Use the previous part, i.e. multiplicative inverses are unique, and observe that $(ab)(a^{-1}b^{-1}) = (aa^{-1})(bb^{-1}) = 1$.]

   c) If $0 < x < y$, then $x^2 < y^2$.

   [Use $0 < x$ and multiply $x < y$ to get $x^2 < xy$. Use $0 < y$ and multiply $x < y$ to get $xy < y^2$. Using the transitive property, gives $x^2 < y^2$.]

5. Negate the statement:

   'For each $\epsilon > 0$ there exists a natural number N such that for every pair $x, y \in [0, 1]$ which satisfies $|x - y| < 1/N$, then $|f(x) - f(y)| < \epsilon$.'

   [There exists $\epsilon > 0$ such that for each natural number N there exists a pair $x, y \in [0, 1]$ which satisfies $|x - y| < 1/N$, but $|f(x) - f(y)| \ge \epsilon$.]

6. a) Prove that the natural numbers are not bounded.

   [See course lecture notes.]

   b) State and prove the Archimedean principle.

   [See course lecture notes.]

   c) Prove that for each $\epsilon > 0$, there exists a natural number N such that for all $N \le n$ there holds $\dfrac{1}{n^2} < \epsilon$.

   [Use the Archimedean Principle to find a natural number N so that $\dfrac{1}{N} < \epsilon$. Notice that if $n \ge N$, then $N \le n \cdot 1 \le n \cdot n$ and so $\frac{1}{n^2} \le \frac{1}{n} \le \frac{1}{N} < \epsilon$]

7. Pick one: Sketch the proof that every open interval (a, b), where $a < b$, contains a rational (irrational) number.

   [See course lecture notes.]

8. a) State the triangle inequality for the real numbers.

   [$|a + b| \le |a| + |b|$]

b) If $|x-3| < \delta \le 1$, then prove that $|x-2| < 2$.
[Represent $(x-2)$ as $(x-3)+1$ and apply the triangle inequality: $|x-2| \le |x-3|+1 \le 1+1 = 2$.]

Extra Credit: If $|x-3| < \delta \le 1$, then prove that $|(x^2-5x+7)-(1)| < 2\delta$.
[$|(x^2-5x+7)-(1)| = |x^2-5x+6| = |(x-3)(x-2)| = |(x-3)| \cdot |(x-2)| \le \delta \cdot 2.$]