More on Compactness

Handout #7, part B

Defn 1. A function f is called *Lipschitz* if there is an $M > 0$ such that

$$|f(x_1) - f(x_2)| \le M|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in dom(f).$$

If $M < 1$, then f is called a *contraction*.

Theorem 1. Each Lipschitz function is uniformly continuous.

Theorem 2. Suppose that K is compact and $f : K \to K$ is a contraction, then f has a fixed point in K.
Proof Let x_0 be an arbitrary point in K. Define inductively,

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots$$

We claim that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent to some $\alpha \in K$. First note that for each $n \in I\!N$

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \le M\,|x_n - x_{n-1}|.$$

Hence, by induction, for each $n \in I\!N$

$$|x_{n+1} - x_n| \le M^n\,|x_1 - x_0|.$$

We then see that if $m > n$, then $m = n + k$ where $k \in I\!N$ and

$$\begin{aligned} |x_{n+k} - x_n| &\le |x_{n+k} - x_{n+k-1}| + |x_{n+k-1} - x_{n+k-2}| + \ldots + |x_{n+1} - x_n| \\ &\le \left(M^{n+k-1} + M^{n+k-2} + \ldots + M^n\right)|x_1 - x_0| \\ &= M^n\,(1 + M + \ldots + M^{k-1})|x_1 - x_0| \\ &\le \tfrac{|x_1 - x_0|}{1-M}\,M^n \end{aligned}$$

and so $\{x_n\}_{n=1}^{\infty}$ is Cauchy. It must converge to some limit α which will belong to K since K is closed. But f is continuous, so $x_{n+1} = f(x_n) \to f(\alpha)$. Notice also that $x_{n+1} \to \alpha$, so α is our fixed point. $\square$

Theorem 3. Suppose that $f : [a, b] \to K$ is one-to-one, onto and continuous, then f^{-1} is continuous.
Proof (#1) Suppose that $g := f^{-1}$ and $y_n \to y_0 \in K$. There exists unique $x_n \in [a, b]$ such that $f(x_n) = y_n$, or equivalently, $x_n = g(y_n)$. If $x_n \not\to x_0$, then there exists $\epsilon_0 > 0$ and a subsequence x_{n_k} such that $|x_{n_k} - x_0| \ge \epsilon_0$. This sequence in turn has

a subsequence which converges in K to some $z \in K$. We may as well assume that the subsequence is the sequence $\{x_{n_k}\}$. f is continuous so $y_{n_k} = f(x_{n_k}) \to f(z)$. But then $f(z) = y_0 = f(x_0)$. f is one-to-one, so $z = x_0$, which is a contradiction, since $|x_{n_k} - x_0| \geq \epsilon_0$. $\square$

Proof (#2) Let $\mathcal{O} \subseteq [a, b]$ be relatively open, then $(f^{-1})^{-1}(\mathcal{O}) = f(\mathcal{O})$. Let C be the complement in $[a, b]$ of $\mathcal{O}$, then C is closed and hence compact. Therefore $f(C)$ is compact in K and consequently it is closed. Its complement in K must then be open. That complement however is $f(\mathcal{O})$. $\square$