Math 554/703 I - Fall 08
Lecture Note Set # 4 - Sequences and Series

Example. The following two results follow from the Principle of Induction and will useful in our study of convergence of sequences and series of real numbers.

1. $\displaystyle\sum_{j=0}^{n} r^j = \frac{1-r^{n+1}}{1-r}$, if $r \neq 1$.
2. $1 + na \leq (1+a)^n$, if $a > 0$ & $n \in I\!\!N$. (Bernoulli's inequality)

Defn. Consider a sequence of points $\{a_n\}$ from a metric space (X, d). The following definitions are used throughout the course:

1. A sequence $\{p_n\}$ in a metric space (X, d) is *convergent to* p, denoted by $\lim_{n\to\infty} p_n = p$, means each ϵ-nbhd of p contains all but a finite number of terms of the sequence. We also use the shorter notation $p_n \to p$ when there is no ambiguity on the indices, or the metric d.
2. $\{p_n\}$ is *bounded* means there is some element $p \in X$ and some real number M for this sequence so that $d(p, p_n) \leq M$, for all $n \in I\!\!N$.
3. $\{p_n\}$ is called *Cauchy* if for each $\epsilon > 0$ there is an $N \in I\!\!N$ so that $d(p_m, p_n) < \epsilon$ whenever $m, n \geq N$.

Example. The following are examples of sequences of real numbers:

1. $1/2, 1/3, 1/4, \cdots$
2. $1, r, r^2, r^3, \cdots$
3. $1,\ 1+r,\ 1+r+r^2,\ 1+r+r^2+r^3, \cdots$

Lemma. $\lim_{n\to\infty} p_n = p$ if and only if

$$\textbf{for every } \epsilon > 0, \textbf{there exists } N \in I\!\!N \textbf{so that if } n \geq I\!\!N, \textbf{then } d(p_n, p) < \epsilon.$$

In short hand this reads '$\forall \epsilon > 0,\ \exists N = N(\epsilon) \in I\!\!N \ni n \geq I\!\!N(\epsilon) \implies d(p_n, p) < \epsilon$.'

Proof. Notice that if a statement is true except for at most a finite number of terms, then there is a largest integer for which it is not true. Take N to be that integer's successor. □

Theorem. If $\lim_{n\to\infty} p_n$ exists, then it is unique.

Proof. Suppose that $\lim_{n\to\infty} p_n = P_1$ and $\lim_{n\to\infty} p_n = P_2$ and that $P_1 \neq P_2$. Set $\epsilon := \frac{1}{2} d(P_1, P_2)$. Now $\epsilon > 0$ so there exists N_1, such that if $n \geq N_1$ then $d(p_n, P_1) < \epsilon$. Since the sequence converges to

P_2, we also have that there exists N_2, such that if $n \geq N_2$ then $d(p_n, P_2) < \epsilon$. Let $N := N_1 + N_2$, then N is larger than both N_1 and N_2 and so

$$d(P_1, P_2) \leq d(P_1, p_N) + d(p_N, P_2) < 2\epsilon = d(P_1, P_2),$$

which gives a contradiction. $\square$

Theorem. Each convergent sequence is bounded.

Proof. Suppose that $\lim_{n\to\infty} p_n = p$. Let $\epsilon := 1$, then there is an integer N such that $p_n \in N_\epsilon(p)$ if $n \geq N$. Set $M := \max\{1, d(p_1, p), d(p_2, p) \ldots, d(p_{N-1}, p)\}$, then the sequence is contained in the neighborhood $N_M(p)$. $\square$

Note. i) In the real numbers a set S is bounded if and only if there exists $M > 0$ so that $|a| \leq M$ for all $a \in S$.

ii) Not every bounded sequence is convergent. For example, the sequence $a_n := (-1)^n$ is bounded, but it is not convergent (take $\epsilon = 1$).

Theorem. Each convergent sequence is Cauchy.

Special Properties of Sequences and Series for $\mathbb{R}$

Examples. The following are important special cases of convergent sequences and series in the metric space $\mathbb{R}$ with the standard metric.

1. $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0.$

   Proof. Use the Archimedean Principle.

2. $\displaystyle\lim_{n\to\infty} \frac{3n^2 - 1}{n^2 + n + 25} = 3.$

   (Hint: Directly - for a given $\epsilon > 0$, use $N := \max\{76, 4N_1\}$ where N_1 is the 'cutoff' for Example 1, i.e. any integer larger than $1/\epsilon$.)

3. If $|r| < 1$, then $r^n \to 0$.

   Proof. If $r = 0$, then the conclusion follows straight away. Suppose that $0 < |r| < 1$, then if $b := 1/|r| - 1$ we see that $b > 0$ and $|r| = 1/(1 + b)$. By Bernoulli's inequality, $|r^n|^{-1} = (1 + b)^n \geq 1 + nb$. Inverting this inequality gives $|r^n - 0| \leq 1/(1 + nb)$. By example 1, pick N so that $1/n < b\epsilon$ if $n \geq N$. Hence,

$$|r^n - 0| \leq \frac{1}{1 + nb} < \frac{1}{nb} < \epsilon, \quad \text{if } n \geq N. \ \square$$

4. $\lim_{n\to\infty} s_n = 1/(1-r)$, if $s_n := 1 + r + r^2 + \cdots + r^n$ and $|r| < 1$.

(Note: $s_n := \sum_{j=0}^{n} r^j$, the sequence of partial sums of the geometric series.)

Proof. If $r = 0$, the conclusion follows immediately. We may suppose then that $0 < |r| < 1$. In this case, we use the identity above, i.e.

$$s_n := \sum_{j=0}^{n} r^n = \frac{1 - r^{n+1}}{1 - r}$$

to see that

$$s_n - s = -r^{n+1}/(1-r)$$

where $s := 1/(1-r)$. Now, given $\epsilon > 0$, by example 3 there is an N_0 such that $n \geq N_0$ implies $|r^n| < (\frac{1-|r|}{|r|})\epsilon$. Combined with the displayed equation, this gives $|s_n - s| < \epsilon$ if $n \geq N_0$. □

Theorem. (Properties of Limits) Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

1. $\lim_{n\to\infty} a_n + b_n = a + b$
2. $\lim_{n\to\infty} a_n b_n = ab$
3. If $b \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Theorem. Suppose that $\lim_{n\to\infty} a_n = a$, then prove that $\lim_{n\to\infty} |a_n| = |a|$.

Defn. A sequence $\{a_n\}$ is called *monotone increasing* if $a_m \leq a_n$ whenever $m \leq n$. A sequence $\{a_n\}$ is called *monotone decreasing* if $a_n \leq a_m$ whenever $m \leq n$.

Theorem. Monotone sequences, which are also bounded, converge.

Theorem. Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = a$. If $a_n \leq c_n \leq b_n$ for all $n \in I\!N$, then $\lim_{n\to\infty} c_n$ exists and equals a.

Theorem. In $\mathbb{R}$, each Cauchy sequence is convergent.

(General metric spaces which have this property are called *complete metric spaces.*)