MATH 554/703 I - FALL 08

Lecture Note Set # 2

Please read Chapter 2 of Rudin which deals with the concepts of countability (pages 24–30) and the metric space topology (pages 30-36). Some portions are outlined again here for convenience.

Defn. a) Two sets are said to have the *same cardinality* and are called *equivalent* if there exists a 1-1, onto mapping from one to the other. This determines an equivalence relation.
b) Consider the set $J_n := \{1, 2, \ldots, n\}$. A set A which is equivalent to J_n, for some n, is called *finite* with cardinality n. This is the *number* of elements of A.
c) A set is called *infinite* if it is not finite.
d) A is called *countable* if it has the same cardinality as the natural numbers $I\!N$.
e) A is said to be *at most countable* if it is countable or finite.

We showed in class that the integers and the rationals were countable, but that the irrationals were not. (We only indicated the proof of the last statement since it needed some elementary properties of sequences which we cover in Lecture Set # 4.) Moreover, we showed ...

Theorem. The union of at most a countable number of at most countable sets is at most countable. (recall to use a diagonal argument)

Theorem. Every infinite subset of a countable set is countable.

Defn. A *sequence* in a set A is a mapping f from the natural numbers to A, which may be written as $\{a_1, a_2, a_3, \ldots\}$ where $a_j := f(j),\ j \in I\!N$.

We recall the concept of *mathematical induction* which is applied to a sequence of statements.

Theorem. (Principle of Mathematical Induction.) Suppose that a statement $\mathbf{p(n)}$ is defined for each natural number n. If

1. $\mathbf{p(1)}$ is true
2. (($\mathbf{p(n)}$ true) $\Longrightarrow$ ($\mathbf{p(n{+}1)}$ true)) is a true statement for each $n \in I\!N$,

then $\mathbf{p(n)}$ is a true statement for each natural number n.

Proof. Suppose to the contrary that the set $B := \{n \in I\!N |\ \mathbf{p(n)} \text{ false}\}$ is not empty. Notice that $1 \notin B$. Let N be the smallest element of B (possible since you can take a minimum of a finite set of integers), then set $n := N - 1$. Observe by assumption (1) that $n \in I\!N$, and by the definition of B that $\mathbf{p(n)}$ is true. By assumption (2), it follows that $\mathbf{p(n{+}1)}$ is true. But $n + 1 = N$. Contradiction. Hence B must be empty. $\square$