Math 554 - Fall 08

Lecture Note Set # 1

Defn. From the introductory lectures, an *ordered set* is a set S with a relation '$<$' which satisfies two properties:

1. (Trichotomy property) for any two elements $a, b \in S$, **exactly one** of the following hold
$$a < b, a = b, \text{ or b < a.}$$
2. (Transitive property) for any three elements $a, b, c \in S$, if $a < b$ and $b < c$, then $a < c$.

In this case the relation '$<$' is called an *order.*

Defn. Suppose that S is an ordered set and $A \subseteq S$. An element $\beta \in S$ is said to be an *upper bound* for A if
$$a \leq \beta, \quad \forall a \in A.$$
An element α is said to be a *least upper bound* for A if

1. α is an upper bound for A
2. if β is any upper bound for A, then $\alpha \leq \beta$.

In this case, the *supremum of A* (=: $\sup A$) is defined as α. The definitions are similar for *lower bound*, *greatest lower bound* and $\inf A$, respectively. Note that we have already shown that the least upper bound (for a nonempty set bounded from above) is unique.

Defn. A set S is said to have the *least upper bound property* if each nonempty subset of S which is bounded from above, has a least upper bound.

Theorem. Suppose the ordered set S has the least upper bound property, then it has the greatest lower bound property (i.e. each nonempty subset of S which is bounded from below has a greatest lower bound).

Proof. Suppose that a nonempty set A has a lower bound, call it ℓ. Define L as the set of all lower bounds of A, then L is nonempty ($\ell \in L$). Observe that each member of the nonempty set A is an upper bound of L so by the least upper bound property, L has a least upper bound. Call this element α.

First observe that α is a lower bound for A. Otherwise, there exists an element $b \in A$ with $b < \alpha$, but each element of A is an upper bound for L, so this element b is an upper bound of L which is smaller than α, the least upper bound of L. This would be a contradiction. Therefore, $\alpha \in L$.

Also, if ℓ is any lower bound of A, then $\ell \leq \alpha$ since $\alpha = \sup L$. Hence α is the greatest lower bound of A. □

Defn. The *real numbers* are defined to be a set $I\!R$ with two binary operations $(+, \cdot)$ which satisfy the following properties: Given any a, b, c in $I\!R$

1. $a + (b + c) = (a + b) + c$.
2. $a + b = b + a$.
3. $\exists 0 \in I\!R \ni a + 0 = a, \forall a \in I\!R$.
4. for each $a \in I\!R$, $\exists(-a) \in I\!R$, so that $a + (-a) = 0$.

5. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
6. $a \cdot b = b \cdot a$.
7. $\exists 1 \in I\!R \ni 1 \neq 0$ and $a \cdot 1 = a, \forall a \in I\!R$.
8. for each $a \in I\!R$ with $a \neq 0$, $\exists a^{-1} \in I\!R$, so that $a \cdot a^{-1} = 1$.
9. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

(Note: These properties just say that $I\!R$ is a nontrivial field.)

Moreover, there is a distinguished subset $I\!P$ (the positive cone) of $I\!R$ with the following properties: Given any a, b in $I\!P$,

a. $a + b \in I\!P$,
b. $a \cdot b \in I\!P$,
c. For each a in $I\!R$, exactly one of the following properties holds:

- **i)** $a \in I\!P$,
- **ii)** $-a \in I\!P$,
- **iii)** $a = 0$.

Finally, $I\!R$ must satisfy the **least upper bound** property, that is, each nonempty subset of $I\!R$ which has an *upper bound* has a *least upper bound*. These terms are defined shortly.

Defn. For the real numbers, define $a < b$ as $b - a \in I\!P$.

Lemma. Using the field properties of $I\!R$, the following properties hold and are all assigned homework problems (see p. 3 of this lecture set):

(1) The additive and multiplicative identities are unique.
(2) The additive and multiplicative inverses are unique.
(3) If $a \in I\!R$, then $a \cdot 0 = 0$.

Theorem. The positive cone $I\!P$ induces an order on the real numbers, i.e. $I\!R$ equipped with the relation '$<$', is an ordered set.

Proof. Assigned homework Problem 2.5. □

Notation:

'$b - a$' is defined as $b + (-a)$.
'$a \leq b$' means either $a < b$ or $a = b$.
The fraction '$\frac{a}{b}$' means $a \cdot b^{-1}$.

Lemma. For each $a, b, c \in I\!R$,

(i.) $(-a) = (-1) \cdot a$.
(ii.) $0 < 1$.
(iii.) if $0 < a$, then $(-a) < 0$.
(iv.) if $a < b$ and $0 < c$, then $a \cdot c < b \cdot c$.
(v.) if $a < b$, then $a + c < b + c$.
(vi.) if $0 < a$, then the multiplicative inverse of a is positive, i.e. $0 < a^{-1}$.
(vii.) the product of two negative real numbers is positive, while the product of a negative real number and a positive real number is negative.

Proof. Since

$$a + ((-1) \cdot a) = (1 \cdot a) + ((-1) \cdot a) = a \cdot (1 + (-1)) = 0 \cdot a,$$

it follows from the fact that $a \cdot 0 = 0$ (Homework Problem 2.4) that $\alpha := (-1) \cdot a$ is an additive inverse for a. But additive inverses are unique (from your Homework Problem 2.2), so the conclusion of part (i) follows.

To prove part (ii), we assume to the contrary, i.e. that $1 \notin \mathbb{P}$. By the definition, $1 \neq 0$, so $(-1) \in \mathbb{P}$ and therefore $0 < -1$. But $(-1) \cdot (-1) = -(-1) = 1$ by part (i) and the HW Problem that additive inverses are unique. This shows that $1 \in \mathbb{P}$ by property (b) of the positive cone and the assumption that $(-1) \in \mathbb{P}$. Contradiction, by the trichotomy property (c).

For part (iii), observe that $0 < a$ means $a \in \mathbb{P}$. Since additive inverses are unique, then $-(-a) = a$, and so $(0 - (-a)) = -(-a) = a \in \mathbb{P}$. This is equivalent to the statement $(-a) < 0$.

To prove (iv), use the definition of $<$ to show both $b - a$ and c are in $\mathbb{P}$. The positive cone is closed under multiplication, so $(b-a)c \in \mathbb{P}$. Using the property (i) shows then that $(bc)-(ac) \in \mathbb{P}$ and so $ac < bc$.

Property (v), is proved similar to showing that $0 < 1$. Indeed, suppose that $b := a^{-1} < 0$, then $-b$ is positive and so $-1 = (-b) \cdot a$ is positive. Contradiction, since -1 is negative.

The proof of property (vi) is left for additional practice. □

Lemma. Suppose that A is a nonempty subset of $\mathbb{R}$, with least upper bound M, then for every $\epsilon > 0$, there exists $a \in A$ such that

$$M - \epsilon < a \leq M.$$

Proof. Since $0 < \epsilon$, then $M - \epsilon < M$. This shows that $M - \epsilon$ cannot be an upper bound for A. Hence there is a member of A, call it a, so that $M - \epsilon < a$. □

Theorem. (Archimedean Property) Suppose a, b are positive real numbers, then there exists $n \in \mathbb{N}$ such that $b < n \cdot a$. (Here $\mathbb{N}$ is the set of *natural numbers*, i.e. 1 and all its *successors*, $1, 1+1, 1+1+1, \ldots$).

Proof. Suppose to the contrary that $na < b$ for all $n \in \mathbb{N}$, then it follows that $\alpha := b/a$ is an upper bound for the natural numbers. Let M be the least upper bound. By the lemma, $1/2 > 0$, so there exists a natural number N so that $M - 1/2 < N$. But then, $M < N + 1/2 < N + 1$, which shows that M is not an upper bound for $\mathbb{N}$. Contradiction. □

Corollary. The natural numbers $\mathbb{N}$ are not bounded.

Corollary. Given any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $0 < 1/n < \epsilon$.

Homework #2 (Due Tuesday Sept 9.)

1. Show that the additive (or multiplicative) identity is unique.
2. Show that additive (or multiplicative) inverses are unique.
3. Prove that $-(a + c) = (-a) + (-c)$.
4. Prove that $a \cdot 0 = 0$ for each a in $\mathbb{R}$.
5. Prove that '$<$' is an order for $\mathbb{R}$.
6. Prove that $a < b$ and $c < 0$ implies that $b \cdot c < a \cdot c$.

7. Prove that if $0 < a < b$, then $0 < \dfrac{1}{b} < \dfrac{1}{a}$.

Notation: Next we define intervals of real numbers.

$(a, b) := \{x \in I\!R | a < x < b\}$ is called the *open interval* with endpoints a, b.
$[a, b] := \{x \in I\!R | a \leq x \leq b\}$ is called the *closed interval* with endpoints a, b.
$(a, b] := \{x \in I\!R | a < x \leq b\}$ and $[a, b) := \{x \in I\!R | a \leq x < b\}$ are called the *half open intervals* with endpoints a, b.
The common length, or measure, of these intervals is defined to be $b - a$.

Theorem. Suppose that I is an interval with endpoints a, b and $a < b$, then I contains a rational number.

Proof. Define the *length of I* by $\ell := b - a$. By the previous corollary, there exists $n_o \in I\!N$ such that $0 < 1/n_o < \ell$. Let $A := \{k |\ k \text{ an integer and } k/n_o < a\}$. A is nonempty, since the negative integers are not bounded from below. Let k_o belong to A. Set $B := \{k |\ k \text{ an integer and } k \geq k_o\} \cap A$. Also, A is bounded from above by $a \cdot n_o$, which shows that B is in fact a finite set of integers. Let K be the largest member of B and therefore of A, then $K + 1 \notin A$. Let $r := (K + 1)/n_o$, then

$$a < \frac{K+1}{n_o} < \frac{K}{n_o} + \ell \leq a + (b - a) = b,$$

which shows that the rational $r \in (a, b) \subseteq I$. □

Corollary. Each interval with nonzero length contains an infinite number of rationals.

Defn. A real number is said to be *irrational* if it is not rational.

Remark: Each interval with nonzero length contains an uncountably infinite number of irrationals. (Proved later.)

We establish a few other facts about irrational numbers and also prove directly that each interval of positive length contains an infinite number of irrationals.

Lemma. The product of a nonzero rational with an irrational is irrational.

Proof. Suppose that $q_1 \cdot \alpha = q_2$, where q_1, q_2 are rational and α is irrational. Since $q_1 \neq 0$, then $\alpha = q_2/q_1$ and it follows that α is rational. Contradiction. □

Lemma. If m is an odd integer, then m^2 is odd.

Proof. If m is odd, then there exists an integer k such that $m = 2k + 1$. In this case $m^2 = 2(2k^2 + 2k) + 1$. □

Lemma. $\sqrt{2}$ is irrational.

Proof. Suppose that $\sqrt{2} = m/n$ where m, n are integers with $n > 0$. We may assume that the rational is in lowest terms (i.e. m and n have no common factors). Squaring the equation and multiplying by n^2, we obtain that $m^2 = 2n^2$. This shows that m^2 is even. By the lemma m must be even and equivalently that it contains 2 as a factor. This shows $4k^2 = 2n^2$ for some integer k.

Consequently, n is even and 2 appears as one of its factors. Contradiction, since m/n was supposed to be in lowest terms. $\square$

Theorem. Each interval with nonzero length contains an infinite number of irrationals.
Proof. Let a, b be the endpoints of the interval I. Consider the interval $(a/\sqrt{2}, b/\sqrt{2})$. It has length $(b-a)/\sqrt{2} > 0$, and so contains a nonzero rational number q. It follows that $q\sqrt{2}$ is between a and b and hence belongs to I. $\square$

Defn. The absolute value of a real number a is defined by

$$|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

Lemma. The *absolute value* function has the following properties:

1. $|a| \geq 0$ for all $a \in \mathbb{R}$.
2. $\pm a \leq |a|$
3. $|-a| = |a|$
4. $|a \cdot b| = |a| \cdot |b|$
5. Suppose $\alpha \geq 0$, then
   $|a| \leq \alpha$ if and only if $-\alpha \leq a \leq \alpha$.
6. $|a + b| \leq |a| + |b|$
7. $||b| - |a|| \leq |b - a|$

Proof. To prove property 1, 'case it out.' Either $a \geq 0$ or $a < 0$. In the first case $|a| = a \geq 0$. In the second case, $|a| = -a > 0$, since $a < 0$. Property 2 follows similarly. Properties 3 and 4 are left for the student.

To prove property 5, notice that $|a| \leq \alpha$ is equivalent to the statement $\pm a \leq \alpha$. Using the elementary order properties, it is easy to see that this last statement is equivalent to both inequalities: $a \leq \alpha$ **and** $-\alpha \leq a$. This is equivalent to the statement on the right hand side of property 5.

To prove property 6, set $\alpha = |a| + |b|$ and apply property 5 after verifying that $\pm(a + b) \leq \alpha$.

To prove property 7, use the fact that $|b| = |(b - a) + a| \leq |b - a| + |a|$ and subtract $|a|$ from each side. This shows that $|b| - |a| \leq |b - a|$. By symmetry in a and b, one proves that $|a| - |b| \leq |a - b| = |b - a|$. $\square$