# Math 554 H.W. #3 Solutions

1. Show that each neighborhood in $\mathbb{R}^K$ is convex.

Solution:

Let $\vec{p} \in \mathbb{R}^K$, $r > 0$, and $N_r(\vec{p}) = \{\vec{x} \in \mathbb{R}^K : |\vec{x} - \vec{p}| < r\}$.

Let $\vec{x}, \vec{y} \in N_r(\vec{p})$ be arbitrary and $0 \le \lambda \le 1$.

Then,

$$
\begin{aligned}
&|\lambda \vec{x} + (1-\lambda)\vec{y} - \vec{p}| \\
&= |\lambda \vec{x} + (1-\lambda)\vec{y} + \lambda \vec{p} + (1-\lambda)\vec{p}| \\
&= |\lambda(\vec{x} - \vec{p}) + (1-\lambda)(\vec{y} - \vec{p})| \\
&\le \lambda |\vec{x} - \vec{p}| + (1-\lambda)|\vec{y} - \vec{p}| \\
&< \lambda r + (1-\lambda) r \\
&= r
\end{aligned}
$$

Therefore, $\lambda \vec{x} + (1-\lambda)\vec{y} \in N_r(\vec{p}) \Rightarrow N_r(\vec{p})$ is convex.

2. Prove that $E'$ is closed.

Proof 1: (Show that $(E')^c$ is open)

Let $x \in (E')^c$ be arbitrary. Then, since $x \notin E'$ $\exists$ $r > 0$ s.t. $N_r(x) \cap E = \emptyset$. We need to show that $N_r(x) \cap E' = \emptyset$. Let $q \in N_r(x)$. Then, since $N_r(x)$ is open $\exists$ $\delta > 0$ s.t. $N_\delta(q) \subset N_r(x) \Rightarrow N_\delta(q) \cap E = \emptyset$. Thus, $q \notin E'$. Since $q \in N_r(x)$ was arbitrary we know that $N_r(x) \cap E' = \emptyset \Rightarrow N_r(x) \subseteq (E')^c$
$\Rightarrow (E')^c$ is open.

Proof 2: (Show that $(E')' \subseteq E'$)

If $(E')' = \emptyset$ we are done. Suppose $(E')' \neq \emptyset$ and let $x \in (E')'$ and $r > 0$ be arbitrary. Then $N_r(x)$ contains infinitely many points in $E'$. Let $q \in N_r(x) \cap E'$ and take $\delta > 0$ s.t $N_\delta(q) \subset N_r(x)$. Since $q \in E'$ $\exists$ infinitely many points in $N_\delta(q) \cap E \Rightarrow$ there are infinitely many points in $N_r(x) \cap E \Rightarrow x \in E' \Rightarrow (E')' \subseteq E'$.

3. Prove that the derived sets of $E$ and of the closure of $E$ coincide.

Proof:

Need to show that $(\overline{E})' = E'$.

Let $x \in E'$ and $r > 0$ be arbitrary. Then, $N_r(x) \cap E$ contains infinitely many points $\Rightarrow N_r(x) \cap \overline{E}$ contains infinitely many points since $E \subseteq E \cup E' = \overline{E}$. Thus, $x \in \overline{E}' \Rightarrow E' \subseteq \overline{E}'$

Let $y \in \overline{E}'$ and $\varepsilon > 0$ be arbitrary. Then, $N_\varepsilon(y) \cap \overline{E} = N_\varepsilon(y) \cap (E \cup E')$ contains infinitely many points. Suppose $N_\varepsilon(y) \cap E'$ has infinitely many points. Then, for $q \in N_\varepsilon(y) \cap E'$ $\exists$ $\delta > 0$ s.t $N_\delta(q) \subseteq N_\varepsilon(y)$ and $N_\delta(q) \cap E$ has infinitely many points since $q \in E'$. Thus, $N_\varepsilon(y) \cap E$ has infinitely many points $\Rightarrow y \in E'$. If $N_\varepsilon(y) \cap E'$ does not have infinitely many points, then $N_\varepsilon(y) \cap E$ must have infinitely many points $\Rightarrow y \in E'$.

Therefore, $\overline{E}' \subseteq E'$.

4. Do $E$ and $E'$ have the same set of limit points?

Answer:

No. Consider the set $E = \{\frac{1}{n} : n \in \mathbb{N}\}$

Then, $E' = \{0\}$ and $(E')' = \emptyset$.

5. Prove that the closure of $E$ is a closed set.

Proof:

From #3 we know that $\bar{E}' = E'$.

Thus, $\bar{E}' = E' \subseteq E \cup E' = \bar{E}$

$\Rightarrow \bar{E}' \subseteq \bar{E}$

$\Rightarrow \bar{E}$ is closed.

(If E closure were not closed mathematicians would have a weird naming convention).

pg. 43 #5. Construct a bounded set in $\mathbb{R}$ that has exactly 3 limit points

Solution:

Consider

$$E = \{\tfrac{1}{n} : n \in \mathbb{N}\} \cup \{3 + \tfrac{1}{n} : n \in \mathbb{N}\} \cup \{5 + \tfrac{1}{n} : n \in \mathbb{N}\}$$

$$\Rightarrow E' = \{0, 3, 5\}$$

pg. 43 #10.

Let $X$ be an infinite set and for $p, q \in X$ define
$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}.$$

Prove that $d$ is a metric. What sets are open? What sets are closed?

Solution:

Let $x, y, z \in X$. Then,

i) $d(x,y) = 0$ or $1 \Rightarrow d(x,y) \geq 0$

ii) $d(x,y) = 0 \Leftrightarrow x = y$ is clear

iii) $d(x,y) = d(y,x)$ is clear

iv) If $x = z$ $d(x,z) = 0 \leq d(x,y) + d(y,z)$

If $x \neq z$, then $x \neq y$ and/or $z \neq y$
$$\Rightarrow d(x,z) \leq d(x,y) + d(y,z).$$

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Every set in $(X,d)$ is both open and closed. For instance, let $E \subseteq X$ be arbitrary. Take $p \in E$ and $r = 1/2$. Then, $N_r(p) = \{p\} \subseteq E$ so $E$ is open. Similarly $E^c$ is open, thus $E = (E^c)^c$ is also closed since its compliment is open.

pg. 43 #11. Show that

a) $d_2(x,y) = \sqrt{|x-y|}$

b) $d_5(x,y) = \dfrac{|x-y|}{1+|x-y|}$

are metrics.

Solution (a):

Let $x, y, z \in \mathbb{R}$.

i) $d_2(x,y) = \sqrt{|x-y|} \geq 0$ is clear

ii) $d_2(x,y) = 0 \Rightarrow \sqrt{|x-y|} = 0 \Rightarrow |x-y| = 0 \Rightarrow x = y$

iii) $d_2(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d_2(y,x)$

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iv)
$$\begin{aligned}
\left(d_2(x,z)\right)^2 &= |x-z| \\
&\le |x-y| + |y-z|, \text{ since } |\cdot| \text{ is a metric} \\
&\le |x-y| + |y-z| + 2\sqrt{|x-y||y-z|} \\
&= \left(\sqrt{|x-y|} + \sqrt{|y-z|}\right)^2 \\
&= \left(d_2(x,y) + d_2(y,z)\right)^2
\end{aligned}$$

$$\Rightarrow d_2(x,z) \le d_2(x,y) + d_2(y,z)$$

Solution (b):

(i)-(iii) are clear.

iv) If $x = z$ then obvious, so assume $|x-z| > 0$.

$$\frac{|x-z|}{1+|x-z|} = \frac{1}{\frac{1}{|x-z|}+1} \le \frac{1}{\frac{1}{|x-y|+|y-z|}+1}$$

$$= \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|}$$

$$= \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|}$$

$$\le \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} = d_5(x,y) + d_5(y,z).$$