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Ronald A. DeVore
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Maximal functions measuring smoothness

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Abstract

Maximal functions which measure the smoothness of a function are introduced and studied from the point of view of their relationship to classical smoothness and their use in proving embedding theorems, extension theorems and various results on differentiation. New spaces of functions which generalize Sobolev spaces are introduced.

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Glossary

Maximal Operators

$f_\alpha^\#$	maximal function based on $P_{[\alpha]}$	(p. 8)
$f_\alpha^\flat$	maximal function based on $P_{(\alpha)}$	(p. 8)
$f_{\alpha,q}^\#$	maximal function based on P	(p. 22)
$f_{\alpha,q}^\flat$	maximal function based on $P^\flat$	(p. 22)
$N_q^\alpha(f,x)$	Calderón maximal operator	(p. 28)
M, M_q	Hardy-Littlewood maximal operators	(p. 9, 23)
$\mathcal{M}_Q, \mathcal{M}_q$	variants of Hardy-Littlewood maximal operators	(p. 9, 23)
F^{**}	the averaged rearrangement of F	(p, 63)

Spaces

$\mathbb{P}_k$	polynomials of total degree at most k	(p. 8)
W_p^k	Sobolev spaces of order k	(p. 17)
$B_p^{\alpha,q}$	Besov spaces of order α	(p. 19)
C_p^α		(p. 36; p. 104 for p < 1)
$\overset{\flat}{C}{}_p^\alpha$		(p. 36; p. 104 for p < 1)
$X_{\theta,q}$; $(X_1,X_2)_{\theta,q}$	"real" interpolation spaces	(p. 65)

Projections

P, P_k	projections from L_1 (unit cube) onto $\mathbb{P}_k$	(p. 8)
P_Q	projections from $L_1(Q)$ onto $\mathbb{P}_k$ induced by P	(p. 8)
f_Q	average of f over Q	(p. 8)
P_Q, $P_Q^\#$	best approximation of degree $[\alpha]$ on $L_q(Q)$	(p. 22)
$P_Q^\flat$	best approximation of degree (α) on $L_q(Q)$	(p. 22)

Glossary

General

$[\alpha]$	greatest integer $\leqq \alpha$	(p. 8)
(α)	greatest integer $< \alpha$	(p. 8)
Q_0	unit cube in $\mathbb{R}^n$	(p. 8)
Ω	open set in $\mathbb{R}^n$	(p. 8)
Δ_h^k	k^{th} difference with step size h	(p. 14)
λQ	dilation of Q by λ	(p. 16)
$\omega_r(f,t)_p$	r^{th} order modulus of smoothness in L_p	(p. 19)
$D_\nu f(x)$	ν^{th} Peano derivative of f at x	(p. 30)
$D^\nu f$	ν^{th} distributional derivative	(p. 33)
P_x	Taylor polynomial	(p. 29, 32)
$K(f,t;X_o,X_1)$	Peetre K-functional	(p. 59)
$K_r(f,t)_p$		(p. 47)
f^*	the decreasing rearrangement of $\|f\|$	(p. 22)
c	generic constant depending at most on α and n unless otherwise specified.	

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To Jana and Carla

§1. Introduction

Maximal functions play a central role in the study of differentiation, singular integrals and almost everywhere convergence. For example, the classical Lebesgue differentiation theorem follows readily from the mapping properties of the Hardy-Littlewood maximal operator:

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f| \tag{1.1}$$

where the sup is taken over all cubes $Q \subset \mathbb{R}^n$ which contain x. The key property of M for differentiation theory is that M is of weak type (1,1), i.e.

$$|\{x: Mf(x) > y\}| \leq \frac{c}{y} \int_{\mathbb{R}^n} |f|, \qquad y > 0. \tag{1.2}$$

It is perhaps less well known that other maximal functions are useful in the study of smoothness of functions and the mapping properties of various operators on smoothness spaces. The main theme of this monograph is to study certain maximal functions of this type and related spaces of functions.

To begin with the simplest example, let $0 \leqq \alpha < 1$ and consider the maximal function

$$f_\alpha^{\#}(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - f_Q| \tag{1.3}$$

where

$$f_Q := \frac{1}{|Q|} \int_Q f \tag{1.4}$$

is the average of f over the cube Q. The maximal function $f_\alpha^{\#}$ was apparently first introduced in a paper of A. P. Calderón and R. Scott [6]. The case $\alpha = 0$ is important in the study of the space BMO - functions of bounded mean oscillation. For example, BMO can be described as the set of functions f such that $f_0^{\#} \in L_\infty$ and $||f_0^{\#}||_{L_\infty}$ is equivalent to the usual BMO norm. The fact that the L_p spaces are interpolation spaces between L_1 and BMO rests on the fact that $f_0^{\#} \in L_p$ is "equivalent" to $f \in L_p$ (see §6).

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When $0 < \alpha < 1$, the maximal function $f_\alpha^{\#}$ measures the smoothness of f. For example if $x, y \in \mathbb{R}^n$, we have the simple inequality (cf. (2.16))

$$|f(x) - f(y)| \leq c\, [f_\alpha^{\#}(x) + f_\alpha^{\#}(y)]\, |x-y|^\alpha .$$

Thus, the finiteness of $f_\alpha^{\#}$ gives a local control for the smoothness of f. In particular, if $f_\alpha^{\#} \in L_\infty$, then $f \in \mathrm{Lip}\,\alpha$ on $\mathbb{R}^n$. Actually, the converse is also true. Namely, if $f \in \mathrm{Lip}\,\alpha$ on $\mathbb{R}^n$ then $f_\alpha^{\#} \in L_\infty$ (see Theorem 6.3).

The mappings $f \to f_Q$ are linear projections from $L_1(Q)$ onto the space of constant functions. They arise from the projection P_0: $f \to \int_{Q_0} f$, $Q_0 = [0,1]^n$, by change of scale. To extend the definition of $f_\alpha^{\#}$ to $\alpha \geq 1$, we replace P_0 by a projection P_k, $k = [\alpha]$, mapping $L_1(Q_0)$ onto $\mathbb{P}_k$ the space of polynomials of degree at most k. Such a projection P gives rise to projections P_Q: $L_1(Q) \to \mathbb{P}_k$ for each Q by change of scale. This leads to the maximal function

$$f_\alpha^{\#}(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - P_Q f|, \qquad P = P_{[\alpha]}. \tag{1.5}$$

It turns out that different projections of the same degree give equivalent maximal functions (see §2). In fact, there is an important property which shows that any projection P of degree $\geq [\alpha]$ when used in (1.5) gives a maximal function equivalent to $f_\alpha^{\#}$ (cf. Lemma 2.3). This is akin to the Marchaud inequalities for moduli of smoothness.

When α is an integer, there is another important, indeed perhaps more natural, choice for the degree of the projection, namely, (α) - the greatest integer strictly less than α. This choice gives the maximal function

$$f_\alpha^{\flat}(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - P_Q f|, \qquad P = P_{(\alpha)}. \tag{1.6}$$

Note that $f_\alpha^{\flat} = f_\alpha^{\#}$ if α is non-integral. Also it can be shown (Corollary [illegible]) that $f_\alpha^{\#} \leq c\, f_\alpha^{\flat}$ if α is an integer.

There are several modifications of the definitions (1.5-6) which lead to equivalent maximal functions. One of the more important is that (§2) the maximal function

$$\sup_{Q \ni x} \inf_{\pi \in \mathbb{P}_k} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - \pi| \tag{1.7}$$

is equivalent to $f_\alpha^\#$ if $k = [\alpha]$ and is equivalent to $f_\alpha^\flat$ if $k = (\alpha)$.

Another important variant is the maximal function defined by

$$N_1^\alpha(f,x) := \sup_{Q\ni x} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f-P_x| \tag{1.8}$$

if there is a polynomial P_x of degree less than α such that (1.8) is finite. Maximal functions of this type were introduced by A. P. Calderón [5] and studied by A. P. Calderón and R. Scott [6]. If there is a P_x which makes (1.8) finite then it is unique. Notice that in (1.8), P_x stays fixed as Q varies, but in (1.6), $P_Q f$ varies with Q. Nevertheless it turns out that the maximal functions $N_1^\alpha(f)$ and $f_\alpha^\flat$ are equivalent (Theorem 5.3). The equivalence of these maximal functions rests on the fact that when $f_\alpha^\flat(x)$ is finite then f has Peano derivatives of order ν at x for each $|\nu| < \alpha$. The polynomial P_x is then the Taylor polynomial of degree (α) formed from these Peano derivatives.

The maximal functions $f_\alpha^\flat$ are related to classical differentiation. For example, it follows from results of Calderón [5] that if $f_k^\flat$ is locally in L_1, then the weak derivatives $D^\nu f$ exist a.e. and satisfy

$$\sum_{|\nu|=k} |D^\nu f(x)| \leq c\, f_k^\flat(x) , \quad \text{a.e.} \tag{1.9}$$

In the other direction, we have

$$f_k^\flat(x) \leq c\, M\Big(\sum_{|\nu|=k} |D^\nu f|\Big)(x) \tag{1.10}$$

whenever f has weak derivatives $D^\nu f$ which are locally in L_1. The connections between the finiteness of the maximal functions $f_\alpha^\#$, $f_\alpha^\flat$ with classical differentiation, Peano derivatives and the like are investigated in §5.

The maximal functions $f_\alpha^\#$, $f_\alpha^\flat$ can be used in a natural way to define new spaces of functions. If $1 \leq p \leq \infty$ and $\alpha > 0$, let $C_p^\alpha := \{f \in L_p : f_\alpha^\# \in L_p\}$ and $||f||_{C_p^\alpha} := ||f||_{L_p} + ||f_\alpha^\#||_{L_p}$. The analogous space $\dot{C}_p^\alpha$ and norm $||\ ||_{\dot{C}_p^\alpha}$ are defined with $f_\alpha^\flat$ in place of $f_\alpha^\#$. These are spaces of smoothness α. The major theme of this work is to study the properties of these spaces and their use in the study of smoothness properties of functions.

There are several smoothness spaces of fractional order. The most useful are the potential spaces $\mathcal{L}_p^\alpha$ (see [15, Chapter V]) and the Besov spaces $B_p^{\alpha,q}$ (see §3). As we have already noted the spaces C_∞^α are related to Lipschitz spaces. Indeed, we have $C_\infty^\alpha = B_\infty^{\alpha,\infty}$ for all $\alpha > 0$. Recall, $B_\infty^{\alpha,\infty}$ is the space Lip α if α is not an integer and is Lip* α (higher order differences) when α is integral. Also $\mathcal{C}_\infty^\alpha$ = Lip α for all $\alpha > 0$. Moreover, it follows from (1.9-10) that $\mathcal{C}_p^k$ is the Sobolev space W_p^k if $1 < p \leqq \infty$ and k is an integer. It turns out that the spaces C_p^α and $\mathcal{C}_p^\alpha$ are not Besov or potential spaces for any other values of p and α. Rather, they offer an attractive alternative to the Besov and potential spaces for many problems in analysis. One of the main advantages of the spaces C_p^α, $\mathcal{C}_p^\alpha$ lies in the fact that for fractional α the function $f_\alpha^\# = f_\alpha^\flat$ is akin to a fractional derivative of f, or better said, a maximal fractional derivative. Thus, these spaces are similar in nature to the Sobolev spaces.

In §7, we establish embeddings between Besov spaces, potential spaces and C_p^α. If $1 \leqq p \leqq \infty$ and $\alpha > 0$, then we have the continuous embeddings

$$B_p^{\alpha,p} \to C_p^\alpha \to B_p^{\alpha,\infty} .$$

These embeddings cannot be improved within the scale of Besov spaces. For potential spaces, we have the continuous embedding

$$\mathcal{L}_p^\alpha \to C_p^\alpha .$$

Of course, $\mathcal{L}_p^\alpha = \mathcal{C}_p^\alpha$ when α is an integer and $1 < p < \infty$ but they are unequal for all other values of p and α.

For fixed $\alpha > 0$, the spaces C_p^α and $\mathcal{C}_p^\alpha$ form interpolation scales as p ranges over $[1,\infty]$. In fact, we show in §8 the characterization of the K functional

$$K(f,t,C_1^\alpha,C_\infty^\alpha) \approx \int_0^t [f^*(s)+f_\alpha^{\#*}(s)]ds \tag{1.11}$$

where g^* denotes the decreasing rearrangement of a function g. A similar result holds for $K(f,t,\mathcal{C}_1^\alpha,\mathcal{C}_\infty^\alpha)$ with $f_\alpha^\#$ replaced by $f_\alpha^\flat$. Of course, (1.11) is a statement about decomposing a function f in C_1^α as f = f-g + g with $g \in C_\infty^\alpha$

and a control on $||f-g||_{C_1^\alpha}$ and $||g||_{C_\infty^\alpha}$. Decompositions of this type were given by A. P. Calderón [5]. For a given $t > 0$, one considers

$$E_t := \{f_\alpha^\# > f_\alpha^{\#*}(t)\} \cup \{Mf > (Mf)^*(t)\}.$$

The function f is smooth outside of E_t. The function g is the extension of f from E_t^c to all of $\mathbb{R}^n$. It is also possible to use the techniques developed for (1.11) to the K functional for interpolation between W_1^k and W_∞^k as was done in R. DeVore-K. Scherer [8]. We should mention that for p fixed and α varying, the spaces C_p^α (or $\dot{C}_p^\alpha$) are not interpolation scales with respect to the real method of interpolation since the corresponding interpolation spaces are Besov spaces [see Theorem 8.6].

We prove Sobolev type embedding theorems for the spaces C_p^α (and $\dot{C}_p^\alpha$) in §9. These follow from inequalities for $f_\alpha^\#$. For example, the inequality

$$(1.12) \qquad |P_Q f(u)-f(u)| \leqq c \int_0^{|Q|} f_\alpha^{\#*}(s)\, s^{\alpha/n} \frac{ds}{s}$$

holds for any Q and f. The right hand side tends to zero as $|Q|\downarrow 0$ whenever $f_\alpha^\# \in L_{n,1}$ (the Lorentz space). This gives the embedding $\{f \in L_1 : f_\alpha^\# \in L_{n/\alpha,1}\} \to C$. The inequality (1.12) (for $\alpha=1$) can be exploited further to give a straight forward proof of the result of E. Stein [16] which says if $\nabla f \in L_{n,1}$ locally then $|f(x+h)-f(x)-\nabla f(x)\cdot h| = o(|h|)$ a.e. in x.

We also establish continuous embeddings $C_p^\alpha \to C_q^\beta$ if $\alpha - \beta = n(1/p-1/q)$ and $1 \leqq p \leqq q \leqq \infty$. In the case $\beta = 0$, the space C_q^β can be replaced by L_q, $1 \leqq q < \infty$ and BMO, $q = \infty$.

Results in the paper are established for domains in $\mathbb{R}^n$. There are two types of results: those that hold for all domains Ω, and those that hold only with some smoothness conditions on Ω. Whenever a result is of the first type, we prove it in its full generality directly. For results of the second type, we establish them originally only for $\Omega = \mathbb{R}^n$ or Ω a cube in $\mathbb{R}^n$. Later in §11, these results are generalized to domains with minimally smooth boundary in the sense of Stein [15] by using extension theorems for the spaces C_p^α and $\dot{C}_p^\alpha$.

We prove the extension theorems of §10-11 using the ideas of Whitney who first proved such extension theorems for Lip α spaces. The constructon uses a Whitney decomposition of Ω^c into cubes $\{Q_j\}$ whose distance to the boundary is comparable to its sidelength and a related partition of unity $\{\phi_j^*\}_1^\infty$ with ϕ_j^* supported on a cube $Q_j^* \subset \Omega^c$ slightly larger than Q_j. Our extension operator then takes the form

$$Ef(x): = \begin{cases} f(x), & x \in \Omega \\ \sum_1^\infty P_{\tilde{Q}_j} f(x)\phi_j^*(x), & x \in \Omega^c \end{cases}$$

where the cubes $\tilde{Q}_j$ are contained in Ω and dist $(\tilde{Q}_j,Q_j) \leqq c$ diam (Q_j). This technique should be compared to the usual approach to extension theorems for Sobolev spaces $W_p^k(\Omega)$ based on potential integrals (see [15, Ch. V]. Since $\mathcal{C}_p^k = W_p^k$, $1 < p \leqq \infty$, our results include extension theorems for Sobolev spaces While preparing this paper, it was pointed out to us by S. Krantz that P. Jones [12] had also used the ideas of Whitney to prove extension theorems for Sobolev spaces although P. Jones' interest is different than ours. Namely he investigates the weakest smoothness on Ω which are sufficient to guarantee extensions for $W_p^k(\Omega)$, $1 \leqq p < \infty$.

In §12, we indicate to what extent the results of the previous sections carry over to the case $p<1$. The spaces C_p^α and $\mathcal{C}_p^\alpha$ for $p<1$ are not defined in terms of $f_\alpha^\#$ and $f_\alpha^\flat$ but instead use variants $f_{\alpha,p}^\#$ and $f_{\alpha,p}^\flat$ which are defined as in (1.7) but with L_p norms in place of L_1 norms. The maximal functions $f_{\alpha,p}^\#$ and $f_{\alpha,p}^\flat$ are studied in §4. We show among other things that for $1 \leqq p \leqq \infty$ the space $\{f \in L_p: f_{\alpha,q}^\# \in L_p\}$ is equal to C_p^α provided that $q \leqq p$. This equivalence only persists for a certain range of $p<1$ and in fact the "proper" definition of C_p^α for $p<1$ is $C_p^\alpha: = \{f \in L_p: f_{\alpha,p}^\# \in L_p\}$. With this definition for example, we have that for fixed α, C_p^α $(p_0 \leqq p \leqq p_1)$ is an interpolation space for the pair $(C_{p_0}^\alpha, C_{p_1}^\alpha)$ whenever $0 < p_0 < p_1 \leqq \infty$. Finally, we indicate the proof of the extension theorem for minimally smooth domains where $0 < p \leqq 1$ and use it to get embedding theorems and interpolation theorems for these domains in this case.

As we have already mentioned, the maximal function f^b_α is equivalent to the maximal function $N^\alpha_1(f)$ introduced by Calderón. For this reason, there is considerable overlap of this work with the papers [5] and [6], most notably in §5 and §8. Rather than refer the readers back to these papers, we have chosen to integrate their results into our development. We have also included some elementary and for the most part well known results about polynomials and approximation in §3.

We have been encouraged by the referee to make some remarks on homogenous spaces. The results presented in this monograph are for non-homogeneous spaces C^α_p, $\mathcal{C}^\alpha_p$. The corresponding homogeneous spaces $\dot{C}^\alpha_p$, $\dot{\mathcal{C}}^\alpha_p$ which are defined as equivalence classes of functions with respect to the seminorms $|\cdot|_{C^\alpha_p}$, $|\cdot|_{\mathcal{C}^\alpha_p}$ are not discussed. These spaces are <u>not</u> merely factor spaces (modulo polynomials of appropriate degree) since the function $f(x) := \phi(x) \log x$ ($\phi \equiv 1$ on (e,∞), $\phi \equiv 0$ on $(-\infty,0)$, smooth otherwise) satisfies $\|f^{\#}_\alpha\|_{L_p} < \infty$ for $p > 1/\alpha > 1$, but $f - \pi$ is not in L_p for any polynomial π. On the other hand, it will be clear to the reader that some of the embeddings of §7 and §9 have analogues for homogeneous spaces. For example, Lemma 2.3 can be modified appropriately to give the analogue of Theorem 9.6 for $\dot{C}^\alpha_p$: If $0 \leqq \beta \leqq \alpha$; $\alpha - \beta = n(\frac{1}{p} - \frac{1}{q})$; $0 < p,q$, then for each $f \in \dot{C}^\alpha_p$ there is a polynomial $\pi \in \mathbb{P}_{[\alpha]}$ so that $|f - \pi|_{C^\beta_q} \leqq c\ |f|_{C^\alpha_p}$. Also the proofs in §7 show that $\dot{B}^{\alpha,p}_p \to \dot{C}^\alpha_p \to \dot{B}^{\alpha,\infty}_p$.

We have included a glossary of notation indicating what the notation means and where it is first introduced or defined. Throughout the paper, we use the symbol c for generic constant whose value may be different at each occurence, even on the same line. <u>Most often</u>, the constant c depends at most on n and α. When this is the case, <u>we will not mention that fact</u>. In all other cases, we shall indicate the quantities on which c depends.

§2. Maximal Functions

Let Q_0 be the unit cube in $\mathbb{R}^n$. The space $\mathbb{P}_k$ of polynomials of (total) degree at most k is a Hilbert space with the inner product $(f,g) := \int_{Q_0} fg$. Consider the orthonormal basis $\{\phi_\nu\}$, $|\nu| \leq k$ which results when the Gram-Schmidt orthogonalization is applied to the power functions $\{x^\nu\}_{|\nu|\leq k}$ arranged in lexicographic order. The operator P defined by

$$Pf := P_k f := \sum_{|\nu|\leq k} (f,\phi_\nu)\phi_\nu \tag{2.1}$$

is a projection from $L_1(Q_0)$ onto $\mathbb{P}_k$.

For any cube Q, the projection P induces a projection P_Q from $L_1(Q)$ onto $\mathbb{P}_k$ by change of scale. In particular when $k = 0$, $P_Q f = f_Q := \frac{1}{|Q|}\int_Q f$. Now take any open set $\Omega \subset \mathbb{R}^n$. If f is locally integrable on Ω and $\alpha \geq 0$, we choose $k := [\alpha]$ and define

$$f^{\#}_\alpha(x) := \sup_{\Omega\supset Q\ni x} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - P_Q f|. \tag{2.2}$$

The maximal function $f^{\#}_\alpha$ measures the smoothness of f. When α is an integer, we have made a choice in (2.2) of taking $k = \alpha$. The choice $k = \alpha-1$ is also important and so we introduce

$$f^{\flat}_\alpha(x) := \sup_{\Omega\supset Q\ni x} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - P^{\flat}_Q f|$$

where $P^{\flat}$ is the projection of degree (α) (the greatest integer strictly less than α). Then $f^{\flat}_\alpha(x) \equiv f^{\#}_\alpha(x)$ if α is not an integer. The study of $f^{\#}_\alpha$, $f^{\flat}_\alpha$ and certain related maximal functions is the main theme of this paper.

There are many variants which can be incorporated into the definition (2.2) while resulting in equivalent maximal functions. From time to time, these variants are more convenient to use in proofs. Therefore, we wish to study some of these possibilities in this section. To this end, we first make some observations about the projections P_Q. It is simple to see by the construction that

(2.3) $$\|P_Q f\|_{L_\infty(Q)} \le c \frac{1}{|Q|} \int_Q |f| .$$

Let x_0 be any point in Q, then there are polynomials h_ν (obtained from fixed polynomials on Q_0 by a change of scale) with $\|h_\nu\|_{L_\infty(Q)} \le c$ for which

(2.4) $$P_Q f(y) = \sum_{|\nu|\le k} \left(\frac{1}{|Q|} \int_Q f\, h_\nu\right) \left[\frac{y - \tilde{x}_0}{|Q|^{1/n}}\right]^\nu$$

where $\tilde{x}_0$ is the point in Q corresponding to x_0 under the change of scale.

Define a Hardy-Littlewood type maximal function (localized to Q) by

(2.5) $$M_Q f(x) := \begin{cases} \sup_{Q \supset \tilde{Q} \ni x} |P_{\tilde{Q}} f(x)|, & x \in Q \\ 0 & , \text{ otherwise} \end{cases}$$

then using (2.3) we see that

(2.6) $$M_Q f(x) \le cM(f\chi_Q)(x) \qquad \text{if } x \in Q$$

where M is the Hardy-Littlewood maximal operator. In particular M_Q is weak-type (1, 1) and strong type (∞, ∞). Moreover, if $x \in \tilde{Q}$,

$$\begin{aligned} |f(x) - P_{\tilde{Q}} f(x)| &\le \|f - f_{\tilde{Q}}\|_{L_\infty(\tilde{Q})} + \|P_{\tilde{Q}}(f - f_{\tilde{Q}})\|_{L_\infty(\tilde{Q})} \\ &\le c\|f - f_{\tilde{Q}}\|_{L_\infty(\tilde{Q})} . \end{aligned}$$

It follows that

(2.7) $$\lim_{\tilde{Q}\downarrow\{x\}} P_{\tilde{Q}} f(x) = f(x)$$

for continuous f. Consequently, the weak type (1, 1) property of the maximal operator M_Q shows that (2.7) holds at each Lebesgue point of f whenever f is in $L_1(Q)$.

Lemma 2.1. If $\alpha \ge 0$ and $k := [\alpha]$, there is a constant $c > 0$ such that

$$c\, f_\alpha^\#(x) \le \sup_{\Omega\supset Q\ni x} \inf_{\pi\in \mathbb{P}_k} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - \pi| \le f_\alpha^\#(x) \qquad x \in \Omega .$$

The same result holds for f_α^b and $k = (\alpha)$.

Proof. The right hand inequality is clear. To prove the left hand estimate, let π be any polynomial of degree at most k. Then $P_Q(\pi) = \pi$

and since P_Q is linear,

$$|f(y) - P_Q f(y)| \leq |f(y) - \pi(y)| + |P_Q(f - \pi)(y)|.$$

Integrating over Q, we obtain from (2.3)

$$\begin{aligned}\int_Q |f - P_Q f|dy &\leq \int_Q |f - \pi|dy + \int_Q |P_Q(f - \pi)|dy \\ &\leq \int_Q |f - \pi|dy + |Q|\ \|P_Q(f - \pi)\|_{L_\infty(Q)} \\ &\leq c \int_Q |f - \pi|dy.\end{aligned}$$

The desired result now follows by taking an infinum over π, dividing by $|Q|^{1+\alpha/n}$, and then taking a supremum over all cubes Q containing x. □

The same proof shows that any other projection $\tilde{P}$ from $L_1(Q_0)$ to $\mathbb{P}_k$ would lead to a maximal function which is equivalent to $f^{\#}_\alpha$. The following is an immediate consequence of the last lemma.

<u>Corollary 2.2</u>. If $\alpha > 0$, there is a constant $c > 0$ such that for each $f \in L_1(\text{loc}\ \Omega)$

$$f^{\#}_\alpha(x) \leq c\ f^{b}_\alpha(x) \quad , \qquad x \in \Omega.$$

The next result shows that the projections P_j with $j > [\alpha]$ (cf. (2.1)) give a maximal function equivalent to $f^{\#}_\alpha$.

<u>Lemma 2.3</u>. If $j \geq [\alpha]$, $\alpha \geq 0$, and

$$F_j(x) := \sup_{\Omega \supset Q \ni x} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - (P_j)_Q f| ,$$

then there are constants c_1, $c_2 > 0$ depending only α, j, and n such that for each $f \in L_1(\Omega) + L_\infty(\Omega)$, $x \in \Omega$

(2.8) $$c_1\ F_j(x) \leq f^{\#}_\alpha(x) \leq c_2\ F_j(x), \qquad \Omega = \mathbb{R}^n$$

(2.9) $$c_1\ F_j(x) \leq f^{\#}_\alpha(x) \leq c_2\ [F_j(x) + \int_\Omega |f|],\ \Omega \text{ the unit cube in } \mathbb{R}^n.$$

Remark: Such upper estimates do not hold for f^{b}_α when α is an integer.

<u>Proof</u>. Using Lemma 2.1, the left hand inequalities in (2.8) and (2.9) are clear since $j \geq [\alpha]$.

For the right hand inequality, let $j > [\alpha]$. We will estimate F_{j-1} by F_j for each such j. Begin by choosing cubes $Q = Q_1 \subset Q_2 \subset \cdots \subset Q_N \subset \Omega$ with $|Q_i| = 2^{-n}|Q_{i+1}|$. Further properties of this sequence will be prescribed shortly. If P denotes the projection operator P_j, we can write

$$f = [f - P_{Q_1} f] + \sum_{i=1}^{N-1} P_{Q_i}(f - P_{Q_{i+1}} f) + P_{Q_N} f \tag{2.10}$$

$$=: f - P_{Q_1} f + \sum_{i=1}^{N-1} \pi_i + \pi_N.$$

Now fix x in Ω. According to (2.4), for $1 \leq i \leq N-1$, each polynomial π_i can be written

$$\pi_i = \sum_{|\nu|=j} [\frac{1}{|Q_i|} \int_{Q_i} (f - P_{Q_{i+1}} f)\, h_{\nu,i}] \left[\frac{y - x}{|Q_i|^{1/n}}\right]^\nu + \rho_i$$

with ρ_i of degree at most j-1. Similary

$$\pi_N = \sum_{|\nu|=j} [\frac{1}{|Q_N|} \int_{Q_N} f\, h_{\nu,N}] \left(\frac{y - x}{|Q_N|^{1/n}}\right)^\nu + \rho_N .$$

Let $\rho := \sum_1^N \rho_i$ so that ρ has degree at most $j - 1$. Using (2.10), (2.3), and the fact that the $h_{\nu,i}$'s are uniformly bounded, we find

$$\frac{1}{|Q|^{1+\alpha/n}} \int_Q |f-\rho| \leq \frac{1}{|Q_1|^{1+\alpha/n}} \int_{Q_1} |f-P_{Q_1} f| \tag{2.11}$$

$$+ \frac{1}{|Q|^{1+\alpha/n}} \sum_{|\nu|=j} \sum_{i=1}^{N-1} [\frac{1}{|Q_i|} \int_{Q_i} |f-P_{Q_{i+1}} f|] \int_Q \left|\left(\frac{y - x}{|Q_i|^{1/n}}\right)^\nu\right| dy$$

$$+ \frac{c}{|Q|^{1+\alpha/n}} \sum_{|\nu|=j} [\frac{1}{|Q_N|} \int_{Q_N} |f|] \int_Q \left|\left(\frac{y - x}{|Q_N|^{1/n}}\right)^\nu\right| dy$$

$$=: I + II + III.$$

We can estimate I trivially

$$I \leq F_j(x).$$

Using the fact that $x \in Q_i \subset Q_{i+1}$ and $|Q_{i+1}| = 2^n |Q_i|$, we find

$$II \leq \frac{c}{|Q|^{1+\alpha/n}} \sum_{i=1}^{N-1} \left[\frac{1}{|Q_i|} \int_{Q_i} |f - P_{Q_{i+1}} f| \right] \left(\frac{|Q|}{|Q_i|}\right)^{j/n} |Q|$$

$$\leq \frac{c}{|Q|^{\alpha/n}} \sum_{i=1}^{N-1} F_j(x)\, |Q_i|^{\alpha/n} \left(\frac{|Q|}{|Q_i|}\right)^{j/n}$$

$$\leq c \left(\sum_{i=1}^{N-1} 2^{i\alpha}\; 2^{-ij} \right) F_j(x) \leq c\, F_j(x)$$

where the constant c does not depend on N.

The sum III can be estimated by

$$\text{(2.12)} \qquad III \leq \frac{c}{|Q_N|} \int_{Q_N} |f| \quad \left(\frac{|Q|}{|Q_N|}\right)^{j/n} |Q|^{-\alpha/n} .$$

If $\Omega = \mathbb{R}^n$, the right hand side tends to 0 as $n \to \infty$. Therefore the estimates for I, II, and III in this case give

$$\text{(2.13)} \qquad \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - \rho| \leq c\, F_j(x).$$

Since ρ is of degree at most $j - 1$, the argument in Lemma 2.1 then shows

$$F_{j-1}(x) \leq c\, F_j(x).$$

Repeated application of this inequality establishes (2.8).

When Ω is the unit cube in $\mathbb{R}^n$, we can choose N so that $Q_N \subset \Omega$ and $2^n|Q_N| > |\Omega|$. In this case (2.12) gives

$$III \leq c \frac{1}{|\Omega|^{j/n+1}} \int_\Omega |f| \quad |Q|^{(j-\alpha)/n} \leq c \int_\Omega |f| .$$

Hence

$$F_{j-1}(x) \leq c[F_j(x) + \int_\Omega |f| \,] .$$

Repeated application of this inequality gives (2.9). □

One other observation will be helpful to us:

(2.14) In the case $\Omega = \mathbb{R}^n$, any maximal function of the type introduced in this section is equivalent to the corresponding maximal function resulting when the supremum over all cubes Q containing the point x is replaced by a supremum over

i) cubes centered at x;

or ii) spheres containing x or centered at x,

or iii) any family of sets $\mathcal{S}_x$ such that for any $S \in \mathcal{S}_x$ there are cubes Q_1, Q_2 containing x with $Q_1 \subset S \subset Q_2$, and $|Q_2| \leq c_0 \; |Q_1|$ where c_0 depends at most on n.

For $f_\alpha^\#$, $f_\alpha^\flat$ this follows from the simple fact that when $S_1 \subset S_2$ and $|S_2| \leq c|S_1|$, then for any $g \geq 0$

$$\frac{1}{|S_1|^{1+\alpha/n}} \int_{S_1} g \;\leq\; \frac{c}{|S_2|^{1+\alpha/n}} \int_{S_2} g.$$

The maximal functions $f_\alpha^\#$ and $f_\alpha^\flat$ give a control for the smoothness of f as will be shown in our next theorem. First we give the following estimate for $P_Q f$. Henceforth, unless otherwise indicated, if $\alpha \geq 0$ then P is the projection operator of degree $[\alpha]$.

Lemma 2.4. If $x \in Q^* \subset Q \subset \Omega$,

$$||D^\nu(P_Q f - P_Q{*}f)||_{L_\infty(Q^*)} \leq c \; |Q|^{\frac{\alpha-|\nu|}{n}} \inf_{u\in Q^*} f_\alpha^\#(u) \tag{2.15}$$

for $0 \leq |\nu| < \alpha$. This inequality also holds for $|\nu| = \alpha$ provided $|Q| \leq 2^n|Q^*|$. The same statements hold for $P^\flat$ replacing P and $f_\alpha^\flat$ replacing $f_\alpha^\#$.

Proof. Consider first the case when $|Q| \leq 2^n|Q^*|$ and $|\nu| \leq \alpha$, then by Markov's inequality

$$||D^\nu(P_Q f - P_{Q^*}f)||_{L_\infty(Q^*)} \leq c|Q^*|^{-|\nu|/n} \; ||P_Q f - P_{Q^*}f||_{L_\infty(Q^*)}.$$

Using (2.3) and the fact that P_Q is a projection gives

$$||P_Q f - P_{Q^*}f||_{L_\infty(Q^*)} \leq ||P_{Q^*}(f - P_Q f)||_{L_\infty(Q^*)} \leq \frac{c}{|Q^*|} \int_{Q^*} |f - P_Q f|$$

$$\leq \frac{c}{|Q|} \int_Q |f - P_Q f| \leq c \; |Q|^{\alpha/n} \inf_{u\in Q^*} f_\alpha^\#(u)$$

which combines with the preceding inequality to give (2.15) in this case.

For the general case of arbitrary $Q^* \subset Q$ and $|\nu| < \alpha$ choose a sequence of nested cubes $Q^* =: Q_1 \subset Q_2 \subset \ldots \subset Q_m \subset Q =: Q_{m+1}$ with $|Q_{i+1}| = 2^n |Q_i|$, $1 \leq i \leq m$, and $|Q_{m+1}| \leq 2^n |Q_m|$, then using the case we have just established, we have

$$\|D^\nu(P_Q f - P_{Q^*} f)\|_{L_\infty(Q^*)} \leq \sum_{i=1}^{m} \|D^\nu(P_{Q_{i+1}} f - P_{Q_i} f)\|_{L_\infty(Q_i)}$$

$$\leq c \sum_{i=1}^{m} |Q_i|^{\frac{\alpha-|\nu|}{n}} \inf_{u\in Q^*} f_\alpha^{\#}(u)$$

$$\leq c|Q|^{\frac{\alpha-|\nu|}{n}} \inf_{u\in Q^*} f_\alpha^{\#}(u)$$

where we have used the fact that $(|Q_i|^{\frac{\alpha-|\nu|}{n}})$ is a geometric sequence. The same proof applies for P^b and f_α^b. □

Let Δ_h denote the difference operator defined by $\Delta_h(f, x) := f(x + h) - f(x)$ and define its powers Δ_h^k inductively as $\Delta_h^k f := \Delta_h(\Delta_h^{k-1} f)$. The difference $\Delta_h^k f$ is defined for each x such that $x, \ldots, x+kh \in \Omega$. Let Ω_h be the set of all points x such that there is a cube $Q_x \subset \Omega$ with $x + ih \in Q_x$, $i = 0, 1, \ldots, k$.

<u>Theorem 2.5</u>. Suppose $k > [\alpha]$ and f is locally integrable on Ω, then for any h,

(2.16) $$|\Delta_h^k(f, x)| \leq c \sum_{i=0}^{k} f_\alpha^{\#}(x + ih)\, |h|^\alpha \quad \text{a.e.} \quad x \in \Omega_h.$$

<u>Proof</u>. Fix h and set $\tilde{\Omega}_h = \{x \in \Omega_h :\ x, \ldots, x + kh$ are Lebesgue points of $f\}$, then $\Omega_h \setminus \tilde{\Omega}_h$ has measure zero. If $x \in \tilde{\Omega}_h$ is fixed, set $y_i := x + ih$ with $i = 0, 1, \ldots, k$. Choose Q as the smallest cube with $\{y_0, y_1, \ldots, y_k\} \subset Q \subset \Omega$. Since each y_i is a Lebesgue point of f, if we choose cubes $Q^* \downarrow \{y_i\}$, then $P_{Q^*} f(y_i) \to f(y_i)$ and so according to Lemma 2.4,

$$|P_Q f(y_i) - f(y_i)| = \lim_{Q^* \downarrow \{y_i\}} |P_Q f(y_i) - P_{Q^*} f(y_i)|$$
$$\leq c\, f_\alpha^{\#}(y_i)\, |Q|^{\alpha/n} .$$

Since $\Delta_h^k(P_Q f) \equiv 0$, we have

$$|\Delta_h^k(f,x)| = |\Delta_h^k(f - P_Q f, x)| \leq c \max_{0\leq i\leq k} |f(y_i) - P_Q f(y_i)|$$
$$\leq c \max_{0\leq i\leq k} f_\alpha^{\#}(y_i)\, |h|^\alpha$$

which gives (2.16). □

§3. Inequalities for Polynomials

In this section, we give several inequalities for polynomials which will be used in the sequel. We begin by comparing various L_q "norms" of polynomials.

Lemma 3.1. If $k \geqq 0$, $q > 0$, there is a constant $c > 0$ depending at most on q, k and n such that for each $q \leqq p \leqq \infty$, each polynomial $\pi \in \mathbb{P}_k$ and each n-cube Q,

$$(3.1) \qquad \left(\frac{1}{|Q|}\int_Q |\pi|^q\right)^{1/q} \leq \left(\frac{1}{|Q|}\int_Q |\pi|^p\right)^{1/p} \leqq c\left(\frac{1}{|Q|}\int_Q |\pi|^q\right)^{1/q} .$$

When either q or $p = \infty$ the corresponding expression is replaced by $||\pi||_{L_\infty(Q)}$

Proof. The left hand inequality is an immediate consequence of Hölder's inequality. It is enough to prove the right hand inequality for $p = \infty$. To this end, choose a point $x_0 \in Q$ such that $|\pi(x_0)| = ||\pi||_{L_\infty(Q)}$. Using Markov's inequality, there is a $c_0 > 0$ depending only on k and n such that

$$|\pi(x) - \pi(x_0)| \leqq |||\nabla\pi|||_{L_\infty(Q)} \, |x-x_0| \leqq c_0 \, ||\pi||_{L_\infty(Q)} \frac{|x-x_0|}{|Q|^{1/n}} .$$

Thus if $S := \{x \in Q: \ |x-x_0| \leqq \frac{|Q|^{1/n}}{2c_0}\}$, then $|S| \geqq c_1 \, |Q|$ with c_1 depending only on n and c_0 and

$$|\pi(x)| \geqq \frac{1}{2} \, ||\pi||_{L_\infty(Q)} , \quad x \in S.$$

Integrating we find

$$||\pi||_{L_\infty(Q)} \leqq 2\left(\frac{1}{|S|}\int_S |\pi|^q\right)^{1/q} \leqq c\left(\frac{1}{|Q|}\int_Q |\pi|^q\right)^{1/q}. \quad \square$$

If Q is an n-cube and $\lambda > 0$, we let λQ denote the cube which has the same center as Q and side length $\lambda\ell(Q)$ where $\ell(Q)$ is the side length of Q.

Lemma 3.2. If $k \geqq 0$; $q, \lambda > 0$, then there is a constant c depending only on k, q, λ and n such that for each $\pi \in \mathbb{P}_k$ and each cube Q, we have

(3.2)
$$\left(\frac{1}{|\lambda Q|}\int_{\lambda Q}|\pi|^q\right)^{1/q} \leq c\left(\frac{1}{|Q|}\int_Q |\pi|^q\right)^{1/q}.$$

In the case $q = \infty$, the norms in (3.2) are replaced by L_∞ norms over λQ and Q respectively.

Proof. For $Q = Q_0$, the unit cube, (3.2) holds for $1 \leq q \leq \infty$ since any two norms on $\mathbb{P}_k$ are equivalent. For any other cube Q and $1 \leq q \leq \infty$, (3.2) now follows from the case Q_0 by a change of variables. The case $q < 1$ follows by using (3.1) with $p = 1$. □

Our next lemma estimates the coefficients of a polynomial.

Lemma 3.3. If $k \geq 0$, $q > 0$, there is a constant c depending only on k, q and n such that for each polynomial $\pi(x) = \sum_{|\nu|\leq k} c_\nu (y-x_0)^\nu$ and any cube Q with $x_0 \in Q$,

(3.3)
$$\sum_{|\nu|\leq k} |c_\nu||Q|^{|\nu|/n} \leq c\left(\frac{1}{|Q|}\int_Q |\pi|^q\right)^{1/q}.$$

When $q = \infty$, (3.3) holds if the right hand side is replaced by $c\,||\pi||_{L_\infty(Q)}$.

Proof. By translating the cube if necessary we can assume $x_0 = 0$. Also in view of (3.1), we need only prove (3.3) for $q = \infty$. When $Q = [-1,1]^n$ and $q = \infty$, then (3.3) follows from the fact that any two norms on $\mathbb{P}_k$ are equivalent. The case $Q = [-\lambda,\lambda]^n$ and $q = \infty$ follows from the case $Q = [-1,1]^n$ by a simple change of variables. Finally, for an arbitrary cube Q of side length ℓ with $0 \in Q$, we have $Q \subset [-\ell,\ell]^n =: \bar{Q}$. Hence

$$\sum_{|\nu|\leq k} |c_\nu|\, \ell^{|\nu|} \leq c\,||\pi||_{L_\infty(\bar{Q})} \leq c\,||\pi||_{L_\infty(Q)}$$

where the last inequality uses (3.2) together with the fact that $\bar{Q} \subset 3Q$. □

We now turn briefly to some well known principles (cf [14] or [7]) concerning the approximation of functions by polynomials. Let $W_p^k(\Omega)$, $1 \leq p \leq \infty$, $k = 1,2,\ldots$, be the Sobolev spaces and

$$|f|_{W_p^k(\Omega)} := \sum_{|\nu|=k} \|D^\nu f\|_{L_p(\Omega)} \tag{3.4}$$
$$\|f\|_{W_p^k(\Omega)} := \|f\|_{L_p(\Omega)} + |f|_{W_p^k(\Omega)} .$$

<u>Theorem 3.4</u>. Let $1 \leqq p \leqq \infty$ and k be a nonnegative integer. There is a constant $c > 0$ depending at most on p, k, n and Ω such that for each cube Q and any $f \in W_p^k(Q)$, there is a polynominal $\pi \in \mathbb{P}_{k-1}$ with

$$\|f-\pi\|_{L_p(Q)} \leqq c\, |Q|^{k/n}\, |f|_{W_p^k(Q)} . \tag{3.5}$$

<u>Proof</u>. It is enough to verify (3.5) for the unit cube Q_0 since the case of arbitrary Q then follows from a linear change of variables. Now suppose (3.5) does not hold for Q_0. In this case, there is a sequence of functions (f_m) such that

$$\inf_{\pi\in\mathbb{P}_{k-1}} \|f_m-\pi\|_{L_p(Q_0)} \geqq m\, |f_m|_{W_p^k(Q_0)} .$$

If we let π_m denote best $L_p(Q_0)$ approximant to f_m, $m = 1,2,\ldots$ then by rescaling if necessary, we find functions $g_m = \lambda_m (f_m-\pi_m)$ such that

$$1 = \inf_{\pi\in\mathbb{P}_{k-1}} \|g_m-\pi\|_{L_p(Q_0)} = \|g_m\|_{L_p(Q_0)} \geqq m\, |g_m|_{W_p^k(Q_0)} .$$

Thus $\{g_m\}_1^\infty$ is precompact in $L_p(Q_0)$ [1, p. 143] and for an appropriate subsequence, $g_{m_j} \to g$ with $g \in L_p(Q_0)$. It follows that

$$|g|_{W_p^k(Q_0)} = \lim_{j\to\infty} |g_{m_j}|_{W_p^k(Q_0)} = 0$$

and so $g \in \mathbb{P}_{k-1}$. On the other hand, $\inf_{\pi\in\mathbb{P}_{k-1}} \|g-\pi\|_{L_p(Q_0)} = 1$ and so we have a contradiction. □

Inequalities like (3.5) hold for more general semi-norms on the right hand side. As another example, we consider the Besov spaces. If Ω is domain in $\mathbb{R}^n$, $|h| > 0$ and r is a positive integer, then define $\Omega_{r,h} := \{x\colon x, x+h,\ldots,x+rh \in \Omega\}$. When $f \in L_p(\Omega)$, $1 \leqq p < \infty$, ($f \in C(\Omega)$ when $p = \infty$), the r-th order modulus of smoothness in $L_p(\Omega)$ is defined by

$$\omega_r(f,t)_p := \sup_{|h|\leq t} \|\Delta_h^r(f)\|_{L_p(\Omega_{r,h})}$$

where Δ_h^r are the usual difference operators (cf. §2).

For any α, $q > 0$, take $r := [\alpha] + 1$ and define

$$(3.6) \qquad |f|_{B_p^{\alpha,q}(\Omega)} := \begin{cases} \{\int_0^\infty [t^{-\alpha}\omega_r(f,t)_p]^q \frac{dt}{t}\}^{1/q} & q < \infty \\ \sup_{0<t} t^{-\alpha}\omega_r(f,t)_p & q = \infty \end{cases}$$

$$\|f\|_{B_p^{\alpha,q}(\Omega)} := \|f\|_{L_p(\Omega)} + |f|_{B_p^{\alpha,q}(\Omega)} .$$

The Besov space $B_p^{\alpha,q}$ is the set of those functions in $L_p(\Omega)$ such that $\|f\|_{B_p^{\alpha,q}(\Omega)}$ is finite. This is a Banach space if $1 \leq p \leq \infty$.

<u>Theorem 3.5</u>. Let $1 \leq p \leq \infty$ and α, $q > 0$. There is a constant $c > 0$ depending at most on p, α, q, n and Ω such that for each n-cube Q and each $f \in B_p^{\alpha,q}(Q)$, there is a $\pi \in \mathbb{P}_{[\alpha]}$ satisfying

$$(3.7) \qquad \|f-\pi\|_{L_p(Q)} \leq c\, |Q|^{\alpha/n}\, |f|_{B_p^{\alpha,q}(Q)} .$$

Remark: The constants in Theorems 3.4 and 3.5 can be chosen independent of p and q but we will not need this.

<u>Proof</u>. Using the fact that the unit ball in $B_p^{\alpha,q}$ is compact in L_p, we can establish (3.7) for $Q = Q_0$ the unit cube in the same way that we have proved (3.6) for the unit cube. For the case of general Q, we note that if f is defined on Q and A is the linear transformation which maps Q_0 onto Q then the function $\tilde{f} = f\circ A$ has a modulus of smoothness which satisfies

$$\omega_r(\tilde{f},t)_p = \ell^{-n/p}\, \omega_r(f,\ell t)_p$$

with ℓ the side length of Q. Thus, $|\tilde{f}|_{B_p^{\alpha,q}(Q_0)} = \ell^{\alpha-n/p}\, |f|_{B_p^{\alpha,q}(Q)}$ and the general case of (3.7) follows easily from the case Q_0. □

We shall need one more technical result which is similar to Theorems 3.4-5 but uses different semi-norms.

Theorem 3.6. Let $0 < k < m$ and $1 \leqq p \leqq \infty$. If Q is a cube in $\mathbb{R}^n$ and $f \in W_p^k(Q)$, then there is a polynomial $\pi \in \mathbb{P}_m$ such that

$$(3.8) \qquad \|f-\pi\|_{L_p(Q)} \leqq c\, |Q|^{k/n} \sum_{|\nu|=k} \Big(\inf_{\pi_\nu \in \mathbb{P}_{m-k}} \|D^\nu f - \pi_\nu\|_{L_p(Q)} \Big)$$

with c depending at most on n, m and p.

Proof. As before, it is enough to prove (3.8) for the unit cube Q_0 since then the case of an arbitrary cube Q follows by change of scale. It is also enough to prove (3.8) for functions f which have a zero polynomial as a best $L_p(Q_0)$ approximation from $\mathbb{P}_m$.

Now suppose (3.8) does not hold for Q_0 and such functions f. Then for each $j = 1,2,\ldots$ there is a function f_j such that

$$(3.9) \qquad \|f_j\|_{L_p(Q_0)} = \inf_{\pi \in \mathbb{P}_m} \|f_j - \pi\|_{L_p(Q_0)} \geqq j \sum_{|\nu|=k} \Big(\inf_{\pi_\nu \in \mathbb{P}_{m-k}} \|D^\nu f_j - \pi_\nu\|_{L_p(Q_0)} \Big) .$$

We can also assume that the f_j have been normalized so that

$$(3.10) \qquad \|f_j\|_{L_p(Q_0)} + \sum_{|\nu|=k} \|D^\nu f_j\|_{L_p(Q_0)} = 1 .$$

It follows that there is a subsequence $(f_{j'})$ of (f_j) such that $f_{j'}$ converges in $L_p(Q_0)$ to a function f in $L_p(Q_0)$.

If $\pi_{\nu,j}$ denotes a best $L_p(Q_0)$ approximation to $D^\nu f_j$ from $\mathbb{P}_{m-k}$, then (3.9) and (3.10) show that for each $|\nu| = k$, $(\pi_{\nu,j})_{j=1}^{\infty}$ is a bounded sequence. Thus we can assume without loss of generality that the subsequence (j') has the property that $\pi_{\nu,j'}$ converges to a polynomial $\pi_\nu \in \mathbb{P}_{m-k}$ for each $|\nu| = k$. It follows from (3.9) that $D^\nu f_{j'}$ converges to π_ν in $L_p(Q_0)$. Also, for any test function $\phi \in C_0^\infty(Q_0)$

$$\int_{Q_0} D^\nu f\, \phi = (-1)^{|\nu|} \int_{Q_0} f\, D^\nu \phi = \lim_{j'\to\infty} (-1)^{|\nu|} \int_{Q_0} f_{j'} D^\nu \phi$$
$$= \lim_{j'\to\infty} \int_{Q_0} D^\nu f_{j'}\, \phi = \int_{Q_0} \pi_\nu \phi .$$

Hence $D^\nu f = \pi_\nu$, $|\nu| = k$. This implies that f is a polynomial in $\mathbb{P}_m$. Since a best approximation to each f_j is the zero polynomial, f also has this property and hence $f \equiv 0$ on Q_0. But this implies $D^{\nu}f \equiv 0$ on Q_0, $|\nu| = k$. This contradicts (3.10) when j' is sufficiently large since $f_{j'} \to f$ and $D^{\nu}f_{j'} \to D^{\nu}f$ ($|\nu| = k$) in $L_p(Q_0)$. □

§4. Additional Estimates

If $0 < q < \infty$, a function f belongs to BMO if and only if

$$\sup_Q \frac{1}{|Q|} \int_Q |f - f_Q|^q < \infty$$

with the supremum taken over all cubes $Q \subset \mathbb{R}^n$. This useful characterization of BMO follows easily from the John-Nirenberg Lemma [10] and is also contained in the inequality [2]

$$(4.1) \qquad [(f - f_Q)\chi_Q]^*(t) \leq c \int_t^{|Q|} (f_Q^{\#})^*(s)\frac{ds}{s}, \qquad 0 < t < \frac{|Q|}{2^n}$$

where $f_Q^{\#}$ denotes the sharp function of f on Q and g^* denotes the decreasing rearrangement of g.

Our interest in this section is to study the analogous situation of taking L_q norms (in place of L_1 norms) in the definition of $f_\alpha^{\#}$ and also to give analogues of (4.1) for $f_\alpha^{\#}$. Such inequalities for $0 < \alpha < 1$ were given in [2]. Throughout this section we assume $\alpha > 0$ unless stated otherwise.

As a starting point, let us introduce some variants of $f_\alpha^{\#}$. If $0 < q < \infty$ and $f \in L_q(Q)$, then f has a set of best approximants from $\mathbb{P}_{[\alpha]}$ in $L_q(Q)$ which we denote by $A(f) := A(f,Q,q)$. Let P_Q be any selection for these best approximants, i.e. $P_Q f \in A(f)$ for each f. Define

$$(4.2) \qquad f_{\alpha,q}^{\#}(x) := \sup_{\Omega \supset Q \ni x} \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|}\int_Q |f - P_Q f|^q\right)^{1/q}$$

$$= \sup_{\Omega \supset Q \ni x} \inf_{\pi \in \mathbb{P}_{[\alpha]}} \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|}\int_Q |f - \pi|^q\right)^{1/q} .$$

Analogously, define

$$(4.3) \qquad f_{\alpha,q}^{b}(x) := \sup_{\Omega \supset Q \ni x} \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|}\int_Q |f - P_Q f|^q\right)^{1/q}$$

where P_Q^b is a selection for best approximation in $L_q(Q)$ by polynomials from $\mathbb{P}_{(\alpha)}$. Note that we can take $P_Q^b = P_Q$ when α is not an integer and therefore $f_{\alpha,q}^{\#} = f_{\alpha,q}^{b}$ for such α. For any $\alpha \geqq 0$,

$$f^{\#}_{\alpha,q} \leq f^{b}_{\alpha,q} . \tag{4.4}$$

As we have shown in Lemma 2.1, $f^{\#}_{\alpha,1}$ is equivalent to $f^{\#}_{\alpha}$ and $f^{b}_{\alpha,1}$ is equivalent to f^{b}_{α}. Actually, for any $q \geq 1$, we can replace $P_Q f$ by $P_Q f$ and get an equivalent maximal function. However for $q < 1$, $P_Q f$ is not necessarily defined since f may not be locally integrable.

Our next task is to give an analogue of the Lebesgue differentiation theorem for $q < 1$. Consider the Hardy-Littlewood type maximal functions

$$M_q f(x) := \sup_{\substack{\Omega \supset Q \ni x \\ \pi \in A(f,Q,q)}} |\pi(x)|$$

where the supremum is taken not only over all cubes containing x but over all best approximants. It is easy to estimate $M_q f$ in terms of the Hardy-Littlewood maximal operator M. Indeed, if $\pi \in A(f,Q,q)$, then

$$\begin{aligned}(\int_Q |\pi|^q)^{1/q} &\leq c\, [(\int_Q |f - \pi|^q)^{1/q} + (\int_Q |f|^q)^{1/q}] \\ &\leq c\, (\int_Q |f|^q)^{1/q} .\end{aligned}$$

Using Lemma 3.1, we have for $x \in Q$,

$$|\pi(x)| \leq ||\pi||_{L_\infty(Q)} \leq c\, (\frac{1}{|Q|} \int_Q |\pi|^q)^{1/q} \leq c\, (\frac{1}{|Q|} \int_Q |f|^q)^{1/q} . \tag{4.5}$$

Taking now a supremum over all π and Q, we find

$$M_q f \leq c\, \mathrm{M}_q f \qquad \text{on } \Omega \tag{4.6}$$

where $\mathrm{M}_q f := [\mathrm{M}(|f|^q)]^{1/q}$ and c depends only on q and α.

The inequality (4.6) shows that M_q is weak type (q,q), i.e. M_q maps L_q into the Lorentz space $L_{q,\infty}$. Using this, we now prove that $P_Q f(x) \to f(x)$ a.e. as $Q \downarrow \{x\}$.

__Lemma 4.1__. If $f \in L_q(\text{loc }\Omega)$, then $\lim_{Q \downarrow \{x\}} P_Q f(x) = f(x)$ a.e. $x \in \Omega$.

__Proof__. Since this is a local result, we may assume that $f \in L_q(\Omega)$. Let

$$Af(x) := \overline{\lim_{Q \downarrow \{x\}}} (\frac{1}{|Q|} \int_Q |f(y) - f(x)|^q dy)^{1/q}.$$

Since $|f| \leq \mathrm{M}_q f$ a.e., we have $Af \leq 2^{1/q}\, \mathrm{M}_q f$ a.e. which shows that A is also weak type (q,q). Therefore

$$|\{x: Af(x) > y\}| \le c\ (\|f\|_{L_q}/y)^q, \qquad y > 0.$$

Now for any continuous function g we have

$$[A(f-g)]^q(x) \le \overline{\lim_{Q\downarrow\{x\}}}\ (\frac{1}{|Q|}\int_Q |f(y)-f(x)|^q dy)$$

$$+ \lim_{Q\downarrow\{x\}}\ (\frac{1}{|Q|}\int_Q |g(y)-g(x)|^q dy)$$

$$= Af(x).$$

Hence, $A(f-g) \le Af$ and it must also follow that $Af \le A(f-g)$ (use $f-g$ in place of f and $-g$ in place of g). We must therefore have $A(f-g) = A(f)$ whenever g is continuous.

Given $\varepsilon > 0$ and $y > 0$ choose g so that $\|f-g\|_{L_q} \le \varepsilon y$, then

$$|\{Af > y\}| = |\{A(f-g) > y\}| \le c\left(\frac{\|f-g\|_{L_q}}{y}\right)^{1/q} \le c\ \varepsilon^{1/q}.$$

Hence $Af = 0$ a.e. and we have shown

$$\lim_{Q\downarrow\{x\}}\ (\frac{1}{|Q|}\int_Q |f(y)-f(x)|^q dy)^{1/q} = 0 \quad \text{a.e.}. \tag{4.7}$$

Return now to $P_Q f$. Fix x_o as any point where (4.7) holds. We have from Lemma 3.1,

$$\|P_Q f - f(x_o)\|_{L_\infty(Q)} \le c\ (\frac{1}{|Q|}\int_Q |P_Q f(y) - f(x_o)|^q dy)^{1/q} \tag{4.8}$$

$$\le c\ (\frac{1}{|Q|}\int_Q |f(y) - f(x_o)|^q dy)^{1/q}$$

where the last inequality uses the fact that $P_q f - f(x_0)$ is a best approximation to $f - f(x_0)$. Taking a limit as $Q\downarrow\{x_0\}$ in (4.8) and using (4.7) shows that $\lim_{Q\downarrow\{x_0\}} P_Q f(x_0) = f(x_o)$. □

Let us now establish our estimates which are similar to (4.1). Notice that if $R^* \subset R$ are cubes with $|R| \le 2^n\ |R^*|$, then

$$\|P_R f - P_{R^*} f\|_{L_\infty(R^*)} \le c(\frac{1}{|R^*|}\int_{R^*} |P_R f - P_{R^*} f|^q)^{1/q}$$

$$\le c[(\frac{1}{|R|}\int_R |f - P_R f|^q)^{1/q} + (\frac{1}{|R^*|}\int_{R^*} |f - P_{R^*} f|^q)^{1/q}] \tag{4.9}$$

$$\le c\ |R^*|^{\alpha/n} \inf_{u\in R^*} f^{\#}_{\alpha,q}(u).$$

Suppose $x \in \Omega$ and $\lim_{Q \downarrow \{x\}} P_Q f(x) = f(x)$. If Q is any cube containing x choose

$Q =: Q_1 \supset \dots \supset Q_j \supset \dots$ with $x \in Q_j$, $j = 1, 2, \dots$, and $|Q_{j+1}| = 2^{-jn}|Q|$,

then using (4.9) we see that

$$(4.10) \qquad |P_Q f(x) - f(x)| \le \sum_{j=1}^{\infty} |P_{Q_j} f(x) - P_{Q_{j+1}} f(x)| \le c\, f^{\#}_{\alpha,q}(x) \sum_{j=1}^{\infty} |Q_j|^{\alpha/n}$$
$$\le c\, |Q|^{\alpha/n} f^{\#}_{\alpha,q}(x)$$

because $x \in Q_j$ for all j. Hence (4.10) holds a.e. on Ω.

The same proofs hold for $f^{b}_{\alpha,q}$ so that

$$(4.9)' \qquad \| P^{b}_{R} f - P^{b}_{R^*} f \|_{L_\infty(R^*)} \le c\, |R^*|^{\alpha/n} \inf_{u \in R^*} f^{b}_{\alpha,q}(u)$$

and

$$(4.10)' \qquad |P^{b}_{Q} f(x) - f(x)| \le c |Q|^{\alpha/n} f^{b}_{\alpha,q}(x) , \quad \text{a.e. } \Omega$$

are valid.

Now we refine the inequalities (4.10), (4.10)' along the lines of (4.1).

<u>Lemma 4.2</u>. If $f \in L_q(\text{loc }\Omega)$, then for each cube $Q \subset \Omega$

$$(4.11) \qquad [(f - P_Q f)\chi_Q]^*(t) \le c\Big[\int_t^{|Q|} F^*(s) s^{\alpha/n} \frac{ds}{s} + t^{\alpha/n} F^*(t)\Big], \quad 0<t\le|Q|/2^n$$

with $F := f^{\#}_{\alpha,q,Q}$ where the subscript Q means that $f^{\#}_{\alpha,q}$ is taken as in (4.2) with Q in place of Ω. The inequality (4.11) holds if P_Q is replaced by P^{b}_{Q} and F is set equal to $f^{b}_{\alpha,q,Q}$.

<u>Proof</u>. Let $E := \{x \in Q: F(x) > F^*(t)\}$ so that $|E| \le t$. If $x \in Q\backslash E$ and $\lim_{Q \downarrow \{x\}} P_Q f(x) = f(x)$, then choose cubes $Q =: Q_1 \supset \dots$ with $|Q_{j+1}| = 2^{-nj}|Q|$ and $x \in Q_j$, $j = 1, 2, \dots$. Let m be the integer with $2^{-(m+1)n} \le \frac{t}{|Q|} < 2^{-mn}$. Using (4.9), we see that

$$(4.12) \qquad |P_Q f(x) - P_{Q_m} f(x)| \le \sum_{2}^{m-1} \| P_{Q_{j-1}} f - P_{Q_j} f \|_{L^\infty(Q_j)} \le c \sum_{2}^{m-1} |Q_j|^{\alpha/n} \inf_{u \in Q_j} F(u)$$
$$\le c \sum_{2}^{m-1} |Q_j|^{\alpha/n} F^*(|Q_j|) \le c \int_t^{|Q|} s^{\alpha/n} F^*(s) \frac{ds}{s}.$$

Since $x \in Q\backslash E$ and $\lim_{j \to \infty} P_{Q_j} f(x) = f(x)$, we have by inequality (4.10) that

$$|P_{Q_m} f(x) - f(x)| \leq c\, F(x)\, |Q_m|^{\alpha/n} \leq c\, F^*(t) t^{\alpha/n}.$$

This inequality combines with (4.12) to show that

$$|P_Q f(x) - f(x)| \leq c\, [\int_t^{|Q_0|} F^*(s) s^{\alpha/n} \frac{ds}{s} + t^{\alpha/n} F^*(t)]$$

outside E. Since $|E| \leq t$, (4.11) follows by the usual properties of decreasing rearrangements. The same proof works for $f^{\flat}_{\alpha,q}$ by using (4.9)' in place of (4.9). □

Using Lemma 4.2 we can now relate $f^{\#}_{\alpha,r}$ and $f^{\#}_{\alpha,q}$. Of course if $q < r$ then it is clear by Hölder's inequality that $f^{\#}_{\alpha,q} \leq f^{\#}_{\alpha,r}$.

<u>Theorem 4.3</u>. If $0 < q < r$ and $f \in L_q(\text{loc}\ \Omega)$, then

$$f^{\#}_{\alpha,r}(x) \leq c\, M_\sigma(f^{\#}_{\alpha,q})(x) \tag{4.13}$$

with $\sigma := (\frac{1}{r} + \frac{\alpha}{n})^{-1}$ and $M_\sigma(g) := [M(|g|^\sigma)]^{1/\sigma}$ where M is the Hardy-Littlewood maximal operator (for Ω). The inequality (4.13) also holds with $\#$ replaced by $\flat$.

<u>Remark</u>. The critical index σ is the smallest value for which $f^{\#}_{\alpha,q} \in L^\sigma$ ensures that $f \in L_r(\text{loc})$. See §9.

<u>Proof</u>. The starting point is inequality (4.11). Applying an L_r norm over Q and using Hardy's inequality [17, p. 196], we obtain

$$\int_0^{|Q|/2^n} [t^{1/r}\psi]^r \frac{dt}{t} \leq c \int_0^{|Q|} [s^{1/\sigma} F^*(s)]^r \frac{ds}{s}$$

where $\psi(t) = [(f - P_Q f)\chi_Q]^*(t)$. But $g = \psi^r$ decreases so $\int_0^{|Q|} g \leq 2^n \int_0^{|Q|/2^n} g$ and consequently

$$(\int_0^{|Q|} [t^{1/r}\psi]^r \frac{dt}{t})^{1/r} \leq c (\int_0^{|Q|} [s^{1/\sigma} F^*(s)]^r \frac{ds}{s})^{1/r}.$$

Given an $x \in Q \subset \Omega$, we divide by $|Q|^{1/\sigma}$ and take a supremum over all $Q \ni x$ in our last inequality to find that

$$f^{\#}_{\alpha,r}(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1/\sigma}} ||f - P_Q f||_{L_r(Q)} \leq \sup_{Q\ni x} \frac{c}{|Q|^{1/\sigma}} ||f^{\#}_{\alpha,q}||_{L^{\sigma,r}(Q)}$$

$$\leq c \sup_{Q\ni x} [\frac{1}{|Q|} \int_Q (f^{\#}_{\alpha,q})^{\sigma}]^{1/\sigma} = c\, M_{\sigma}(f^{\#}_{\alpha,q})(x)$$

where we have used the notation $L_{\sigma,r}$ for the Lorentz norms and the well known inequality $||.||_{L_{\sigma,r}} \leq c\, ||.||_{L_{\sigma,\sigma}} = c||.||_{L_{\sigma}}$ when $\sigma < r$ (see [17, p. 192]). The same proof works for $\flat$ in place of $\#$. □

The following extends Lemma 2.3 to the case $q < 1$.

Lemma 4.4. Let $0 < q < 1$ and $F_j(x) := \sup_{Q\ni x} \frac{1}{|Q|^{\alpha/n}} \inf_{\pi\in \mathbb{P}_j} (\frac{1}{|Q|} \int_Q |f-\pi|^q)^{1/q}$. If $\alpha \geq 0$ and $j \geq [\alpha]$, there is $c_1 > 0$ depending at most on α, j, q and n such that for each $f \in L_q + L_{\infty}$

(4.14) $$F_j(x) \leq f^{\#}_{\alpha,q}(x) \leq c_1 F_j(x) .$$

Proof. The proof is the same as Lemma 2.3 except for certain modifications necessitated by the fact that $q < 1$. The lower inequality in (4.14) follows from the fact that $F_{[\alpha]} = f^{\#}_{\alpha,q}$. The upper inequality follows from the inequality

(4.15) $$F_{j-1}(x) \leq c\, F_j(x),$$

which holds for all $j > [\alpha]$.

To prove (4.15), choose cubes $Q = Q_1 \subset Q_2 \subset \ldots \subset Q_N$ as in Lemma 2.3 and write

$$f = f - P_{Q_1} f + \sum_{i=1}^{N-1} [P_{Q_i} f - P_{Q_{i+1}} f] + P_{Q_N} f = f - P_{Q_1} f + \sum_{1}^{N-1} \pi_i + \pi_N$$

where P is the best L_q projection operator of degree j. We write

$$\pi_i = \rho_i + \text{terms of order } j$$

with ρ_i of degree $\leq j - 1$. If $\rho := \sum_{1}^{N} \rho_i$, then

$$\frac{1}{|Q|} \int_Q |f - \rho|^q \leq I + II + III$$

with the notation corresponding to that in Lemma 2.3.

Each of the terms I, II $\leqq c\,[F_j(x)\ |Q|^{\alpha/n}]^q$. The proof of I is the same as in Lemma 2.3. The proof of II uses Lemma 3.3 and the subadditivity of $\int |\cdot|^q$ with the same basic argument as in Lemma 2.3. Since III $\to 0$ as $N \to \infty$, (4.15) follows. □.

§5. The Calderón Maximal Operator and Peano Derivative

A.P. Calderón [5] and later Calderón and R. Scott [6] have introduced certain maximal operators in conjunction with the study of singular integrals, differentiation and the embeddings of Sobolev spaces. In this section, we shall show that these maximal operators are equivalent to $f^{b}_{\alpha,q}$ and in the process bring out connections between the finiteness of f^{b}_{α} (or $f^{\#}_{\alpha}$) and the differentiability of f.

For $q, \alpha > 0$ and $f \in L_q(\text{loc})$,

define

$$(5.1) \qquad N^{\alpha}_{q}(f,x) := \sup_{\Omega \supset Q \ni x} \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|} \int_Q |f - P_x|^q\right)^{1/q}$$

if there is a polynomial P_x of degree less than α such that (5.1) is finite, otherwise let $N^{\alpha}_{q}(f,x) := +\infty$. This is in essence the maximal function defined by Calderón although Calderón makes the definition only for $q \geqq 1$ ($q > 1$ in [5] and $q \geqq 1$ in [6]) and takes the sup over balls rather than cubes (which as was noted in §2 leads to an equivalent maximal function). It should be emphasized that in contrast to the definitions of $f^{\#}_{\alpha}$ and f^{b}_{α}, the polynomial in (5.1) does not vary with Q. Nevertheless it turns out that $N^{\alpha}_{q}(f)$ and $f^{b}_{\alpha,q}$ are equivalent.

Much of the material of this section can be found in the paper of Calderón [5]. We begin by showing that $N^{\alpha}_{q}(f,x)$ is well defined for each $0 < q < \infty$ and $0 < \alpha$.

Lemma 5.1. If there is a polynomial P_x of degree less than α such that

$$\sup_{\Omega \supset Q \ni x} \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|} \int_Q |f - P_x|^q\right)^{1/q} < \infty,$$

then P_x must be unique.

Proof. Suppose that π_1, π_2 are two polynomials in $P_{(\alpha)}$ which satisfy

$$\sup_{\Omega \supset Q \ni x} \frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|} \int_Q |f - \pi_j|^q\right)^{1/q} < \infty \qquad j = 1,2,$$

then the polynomial $\rho(y) = \pi_1(y) - \pi_2(y) =: \sum_{|\nu|<\alpha} c_\nu (y - x)^\nu$ satifies

$$\left(\frac{1}{|Q|} \int_Q |\rho|^q\right)^{1/q} \leq c \sum_{j=1}^{2} \left(\frac{1}{|Q|} \int_Q |f - \pi_j|^q\right)^{1/q} \leq c\, |Q|^{\alpha/n}$$

for all Q containing x. Because of Lemma 3.3,

$$\sum_{|\nu|<\alpha} |c_\nu|\, |Q|^{|\nu|/n} \leq c\, |Q|^{\alpha/n}$$

for all Q containing x. Letting $|Q| \to 0$ shows that $c_\nu = 0$ for all ν. □

We start with a definition of the ν-th Peano derivative of f at x_0. Suppose there is a $q > 0$ and an open set $\mathcal{O} \subset \Omega$ with $x_0 \in \mathcal{O}$ such that f is in L_q on $\mathcal{O}$. Suppose further there is a family of polynomials $\{\pi_Q\}_Q$ with $x_0 \in Q$ and $\deg \pi_Q \leq M$, for all $Q \subset \mathcal{O}$, and

$$\left(\frac{1}{|Q|} \int_Q |f - \pi_Q|^q\right)^{1/q} = O(|Q|^{k/n}).$$

Then, if $|\nu| < k$

$$\lim_{Q \downarrow \{x_0\}} D^\nu \pi_Q(x_0) =: D_\nu f(x_0) \tag{5.2}$$

exists and is finite. Indeed, when $Q^* \subset Q$ and $|Q^*| \geq 2^{-n}|Q|$, then using Markov's inequality and Lemma 3.1

$$\begin{aligned} \|D^\nu(\pi_Q - \pi_{Q^*})\|_{L_\infty(Q^*)} &\leq c|Q|^{-|\nu|/n} \|\pi_Q - \pi_{Q^*}\|_{L_\infty(Q)} \\ &\leq c|Q|^{-|\nu|/n} \left(\frac{1}{|Q|} \int_Q |\pi_Q - \pi_{Q^*}|^q\right)^{1/q} \\ &\leq c|Q|^{(k-|\nu|)/n} . \end{aligned}$$

Hence, the same exact telescoping argument as used in the proof of Lemma 2.4 shows that for any $x_0 \in Q^* \subset Q$

$$\|D^\nu(\pi_Q - \pi_{Q^*})\|_{L_\infty(Q^*)} \leq c|Q|^{(k-|\nu|)/n} \tag{5.3}$$

which shows that (5.2) exists.

Whenever such a family of polynomials exist, we call $D_\nu f(x_0)$ as defined by (5.2) the <u>ν-th</u> <u>Peano</u> <u>derivative</u> of f at x_0.

Let us observe that $D_\nu f(x_0)$ does not depend on the neighborhood $\mathcal{O}$, the family π_Q, or on q. If $\{\pi_Q\}$ is a family for $\mathcal{O}$, q, and k, and $\{\tilde{\pi}_Q\}$ a family for $\tilde{\mathcal{O}}$, $\tilde{q}$, and $\tilde{k}$, it follows for a suitably chosen $\mathcal{O}_0$, that whenever $Q \subset \mathcal{O}_0$ and q_0 is the minimum of q and $\tilde{q}$,

$$||\pi_Q - \tilde{\pi}_Q||_{L_\infty(Q)} \leqq c\left(\frac{1}{|Q|}\int_Q|\pi_Q - \tilde{\pi}_Q|^{q_0}\right)^{\frac{1}{q_0}}$$

$$\leqq c\left[\left(\frac{1}{|Q|}\int_Q|\pi_Q - f|^q\right)^{\frac{1}{q}} + \left(\frac{1}{|Q|}\int_Q|\tilde{\pi}_Q - f|^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}}\right]$$

$$\leqq c|Q|^{k_0/n}$$

with k_0 the minimum of k and $\tilde{k}$. Since $|\nu| < k_0$

$$|D^\nu(\pi_Q - \tilde{\pi}_Q)(x_0)| \leqq c|Q|^{-|\nu|/n}(|Q|^{k_0/n}) = o(1) .$$

This shows that $\lim_{Q\downarrow\{x_0\}} D^\nu\pi_Q(x_0) = \lim_{Q\downarrow\{x_0\}} D^\nu\,\tilde{\pi}_Q\,(x_0)$.

<u>Lemma 5.2</u>. If α, $q > 0$; $|\nu| < \alpha$ and f is locally in L_q, then $D_\nu f(x)$ exists at each point where $f^{\#}_{\alpha,q}(x)$ is finite. In addition, for such x

$$(5.4) \qquad |D^\nu(P_Q f)(x) - D_\nu f(x)| \leqq c\ f^{\#}_{\alpha,q}(x)\ |Q|^{\frac{\alpha-|\nu|}{n}} .$$

If $f^{b}_{\alpha,q}(x)$ is finite, then

$$(5.4)' \qquad |D^\nu(P^b_Q f)(x) - D_\nu f(x)| \leqq c\ f^{b}_{\alpha,q}(x)\ |Q|^{\frac{\alpha-|\nu|}{n}} .$$

<u>Proof</u>. If $x \in R_2 \subset R_1$ and $|R_2| \geqq 2^{-n}\ |R_1|$, then from (4.9)

$$||P_{R_1}f - P_{R_2}f||_{L_\infty(R_2)} \leqq c\ f^{\#}_{\alpha,q}(x)\ |R_2|^{\alpha/n} .$$

Using the exact same telescoping argument as in Lemma 2.4 shows that

$$(5.5) \qquad ||D^\nu(P_Q f - P_{Q^*}f)||_{L_\infty(Q^*)} \leqq c\ f^{\#}_{\alpha,q}(x)\ |Q|^{(\alpha-|\nu|)/n}$$

for any cubes Q, Q^* with $x \in Q^* \subset Q$. Hence $\{P_Q f\}$ can be used in (5.2), and so $\lim_{Q\downarrow\{x\}} D^\nu\ P_Q f(x) = D_\nu f(x)$. Using this in (5.5) gives (5.4). To prove (5.4)' use (4.9)' in place of (4.9) and P^b_Q in place of P_Q in the above argument. □

Theorem 5.3. If α, $q > 0$, there are constants c_1, $c_2 > 0$ such that for each $f \in L_q(\mathrm{loc})$,

$$c_1\, f^\flat_{\alpha,q}(x) \leq N^\alpha_q(f,\, x) \leq c_2\, f^\flat_{\alpha,q}(x), \qquad x \in \Omega \,. \tag{5.6}$$

Proof. The lower estimate in (5.6) is clear from the definitions of these maximal functions. For the upper estimate, suppose $f_{\alpha,q}(x)$ is finite and define $P_x(y) := \sum_{|\nu|<\alpha} D_\nu f(x) \frac{(y-x)^\nu}{\nu!}$ where $D_\nu f(x)$ are the Peano derivatives of f at x which are guaranteed to exist by Lemma 5.2. Using (5.4)', we find for any cube $Q \ni x$

$$\left(\frac{1}{|Q|} \int_Q |P_x - P^\flat_Q f|^q\right)^{1/q} \leq c\, \|P_x - P^\flat_Q f\|_{L_\infty(Q)}$$

$$\leq c \sum_{|\nu|<\alpha} |D_\nu f(x) - D^\nu P^\flat_Q f(x)|\ \|(\cdot - x)^\nu\|_{L_\infty(Q)}$$

$$\leq c \sum_{|\nu|<\alpha} f^\flat_{\alpha,q}(x)\ |Q|^{\frac{\alpha-|\nu|}{n}}\ |Q|^{\frac{|\nu|}{n}} \leq c\, f^\flat_{\alpha,q}(x)\ |Q|^{\alpha/n}.$$

But $(\int_Q |f - P_x|^q) \leq c\,(\int_Q |f - P^\flat_Q f|^q + \int_Q |P_x - P^\flat_Q f|^q)$ which together with the last inequality shows $\left(\frac{1}{|Q|}\int_Q |f - P_x|^q\right)^{1/q} \leq c\, f^\flat_{\alpha,q}(x)\ |Q|^{\alpha/n}$. Dividing by $|Q|^{\alpha/n}$ and taking a supremum over all Q establishes the right hand inequality in (5.6). □

Corollary 5.4. If $\alpha > 0$ there are constants c_1, $c_2 > 0$ such that

$$c_1\, f^\flat_\alpha(x) \leq N^\alpha_1(f,x) \leq c_2\, f^\flat_\alpha(x).$$

Proof. $f^\flat_{\alpha,1}(x)$ is equivalent to $f^\flat_\alpha$ because of Lemma 2.1. □

Corollary 5.5. If $f^\flat_{\alpha,q}(x) < \infty$, then $P_x(y) = \sum_{|\nu|<\alpha} D_\nu f(x) \frac{(y-x)^\nu}{\nu!}$.

Proof. This follows immediately from the proof of Theorem 5.3 and the uniqueness of P_x. □

When α is an integer $f^\flat_\alpha$ can be estimated in terms of classical derivatives as the following result shows.

Theorem 5.6. There are constants c_1, $c_2 > 0$ such that for any $f \in W_1^k(\text{loc } \Omega)$

$$f_k^b(x) \leq c_1 M\Big(\sum_{|\nu|=k} |D^\nu f|\chi_\Omega\Big)(x) \tag{5.7}$$

and for any $f \in L_1(\text{loc})$ for which $f_k^b \in L_1(\text{loc})$, the weak derivatives $D^\nu f$, $|\nu| = k$, exist and satisfy

$$\sum_{|\nu|=k} |D^\nu f(x)| \leq c_2 f_k^b(x) \qquad \text{a.e. } x \in \Omega. \tag{5.8}$$

Proof. Let $\mathcal{D}f := \sum_{|\nu|=k} |D^\nu f|$. When $f \in W_1^k(\text{loc } \Omega)$ and Q is a cube contained in Ω, then according to Theorem 3.4 there is a polynomial π of degree $< k$ with

$$\int_Q |f - \pi| \leq c\, |Q|^{k/n} \int_Q |\mathcal{D}f|.$$

Dividing by $|Q|^{k/n+1}$ and taking an inf over π and a sup over all Q containing x gives (5.7).

To prove (5.8), let $f \in L_1(\text{loc})$ and consider any test function $\phi \in C_0^\infty(\Omega)$ with supp $\phi =: K \subset\subset \Omega$. Choose a function $\psi \in C^\infty$ with ψ supported on the unit cube and $\int_{\mathbb{R}^n} \psi = 1$. Set $\psi_\varepsilon(x) := \varepsilon^{-n} \psi(\varepsilon^{-1}x)$. If $\varepsilon > 0$ is sufficiently small the functions $F_\varepsilon := f * \psi_\varepsilon$ are defined on K. Also for any $|\nu| = k$, we have for $z \in K$

$$\begin{aligned} |D^\nu F_\varepsilon(z)| &= \Big|\int_{\mathbb{R}^n} f(y) D^\nu \psi_\varepsilon(z-y)dy\Big| = \Big|\int_{\mathbb{R}^n} [f(y) - P_z(y)] D^\nu \psi_\varepsilon(z-y)dy\Big| \\ &\leq c\, \varepsilon^{-k-n} \int_{z+Q_\varepsilon} |f(y) - P_z(y)|dy \leq c\, N_1^k(f,z) \leq c\, f_k^b(z) \end{aligned} \tag{5.9}$$

with Q_ε the cube with side length 2ε centered at 0. The second equality uses the fact that $\int P\, D^\nu g = (-1)^{|\nu|} \int D^\nu P\, g = 0$ if g has compact support and P is a polynomial of degree less than $|\nu|$. We also used the fact that $\|D^\nu \psi_\varepsilon\|_\infty \leq \varepsilon^{-k-n} \|D^\nu \psi\|_\infty \leq c\, \varepsilon^{-k-n}$ and ψ_ε is supported on Q_ε. The last inequality is Corollary 5.4.

Using (5.9), we have

$$\Big|\int_{\mathbb{R}^n} D^\nu \phi\, f\Big| = \lim_{\varepsilon \to 0} \Big|\int_K D^\nu \phi\, F_\varepsilon\Big| \leq \lim_{\varepsilon \to 0} \int_K |\phi|\, |D^\nu F_\varepsilon| \leq c \int |\phi| f_k^b$$

This estimate shows that the distributional derivative $D^\nu f$ is a distribution of order 0 and hence must be a Radon measure. Moreover, the same estimates show that $D^\nu f$ must be absolutely continuous with respect to Lebesgue measure. Therefore $D^\nu f$ must belong to $L_1(\mathrm{loc})$ and satisfy

$$|D^\nu f| \leqq c\, f_k^b \quad \text{a.e.}$$

as desired. □

Remark. The preceding proof actually shows that the weak derivatives $D^\nu f$ ($|\nu| = k$) exist and satisfy

$$|D^\nu f(z)| \leqq c\, F_k(z) := \sup_{\substack{\Omega\supset Q\ni x\\ |Q|\leqq 1}} \left(\frac{1}{|Q|^{1+k/n}} \int_Q |f(y) - P_z(y)|\right) \tag{5.10}$$

whenever F_k is locally integrable. This follows since the integration in inequality (5.9) was performed over cubes of measure $(2\varepsilon)^n$ as $\varepsilon \to 0+$.

The following Corollary extends Theorem 5.6 to the case of nonintegral α.

Corollary 5.7. Suppose $\alpha > 0$ and $f_\alpha^b \in L_1(\mathrm{loc}\ \Omega)$, then for each $|\nu| < \alpha$ both the weak derivatives $D^\nu f$ and the Peano derivatives $D_\nu f$ exist a.e., are locally integrable, and coincide a.e. on Ω. Moreover,

$$|D^\nu f(x)| \leqq c\, [f_\alpha^b(x) + \textstyle\int_Q |f| \,/\, |Q|^{1+|\nu|/n}] \quad \text{a.e. } \Omega \tag{5.11}$$

where Q is any cube satisfying $|Q| \leqq 1$ and $\Omega \supset Q \ni x$.

Proof. Let $k = (\alpha)$ and suppose $|\nu| \leqq k$, then according to Lemma 5.2 the Peano derivative $D_\nu f$ exists a.e. and satisfies for any cube Q with $x \in Q \subset \Omega$

$$\begin{aligned} |D_\nu f(x)| &\leqq c\, f_\alpha^b(x)|Q|^{\frac{\alpha-|\nu|}{n}} + |D^\nu(P_Q f)(x)| \\ &\leqq c[|Q|^{\frac{\alpha-|\nu|}{n}} f_\alpha^b(x) + \frac{1}{|Q|^{1+|\nu|/n}} \textstyle\int_Q |f|] \end{aligned} \tag{5.12}$$

where the last inequality follows from the representation of $P_Q f$ given in (2.4).

Next we prove that the weak derivaties are locally integrable. Suppose $|\nu| = k$ and let F_k denote the maximal function defined in (5.10). Since the supremum in (5.10) is over all cubes Q with $|Q| \leqq 1$, it follows from Corollary 5.4 that

(5.13) $$F_k(x) \leqq N_1^\alpha(f,x) \leqq c\, f_\alpha^b(x).$$

Since F_k is locally integrable, inequality (5.10) shows that $D^\nu f$ is also locally integrable. Hence, as is well known [1, p. 75], $D^\mu f$ is locally integrable for each $|\mu| < k$.

Finally, in order to complete the proof of the theorem, we must show that $D^\nu f = D_\nu f$ a.e. on Ω for $|\nu| \leqq k$. Define $P_x(y) := \sum_{|\nu|\leqq k} D_\nu f(x) \frac{(y-x)^\nu}{\nu!}$. Let $\psi \in C^\infty$ be supported on the unit cube with $\int\psi = 1$ and set $\psi_\varepsilon(x) := \varepsilon^{-n}\, \psi(\varepsilon^{-1}x)$, $\varepsilon > 0$. If Q is any closed cube contained in Ω, then $D^\nu f * \psi_\varepsilon$ is defined on Q provided ε is sufficiently small. Moreover (see [15, p. 62]),

(5.14) $$D^\nu f(x) = \lim_{\varepsilon\to 0^+} D^\nu f * \psi_\varepsilon(x) = \lim_{\varepsilon\to 0^+} f * D^\nu\psi_\varepsilon(x), \quad \text{a.e.} \quad x \in \Omega.$$

Let x be any point in Q where (5.14) holds and where both $D_\nu f(x)$ and $D^\nu f(x)$ exist. Since P_x is a polynomial, $\lim_{\varepsilon\to 0^+} D^\nu P_x * \psi_\varepsilon(y) = D^\nu P_x(y)$ holds for each y. But $D^\nu(P_x)(x) = D_\nu f(x)$ by the definition of P_x, so

$$\begin{aligned} |D^\nu f(x) - D_\nu f(x)| &= |D^\nu f(x) - D^\nu P_x(x)| \\ &= \lim_{\varepsilon\to 0^+} |(f-P_x) * D^\nu\psi_\varepsilon(x)| \\ &\leqq \overline{\lim_{\varepsilon\to 0^+}}\, \varepsilon^{-n-|\nu|} \int_{|y-x|\leqq\varepsilon} |f(y)-P_x(y)|\,dy \\ &\leqq c\, \overline{\lim_{\varepsilon\to 0^+}}\, f_\alpha^b(x)\, \varepsilon^{\alpha-|\nu|} = 0. \end{aligned}$$

The last inequality follows since $N_1^\alpha(f,x) \leqq c\, f_\alpha^b(x)$ by Corollary 5.4. □

§6. Smoothness Spaces

We have already pointed out that $f_\alpha^\sharp$ measures the local smoothness of f. Accordingly for $1 \leqq p \leqq \infty$ [see §12 for the case $0 < p < 1$] and $\alpha > 0$, we define smoothness spaces

$$C_p^\alpha \ := \ C_p^\alpha(\Omega) : = \{f \in L_p(\Omega) : f_\alpha^\sharp \in L_p(\Omega)\}$$

and

$$\dot{C}_p^\alpha := \{f \in L_p(\Omega) : f_\alpha^\flat \in L_p(\Omega)\},$$

then $\dot{C}_p^\alpha \subset C_p^\alpha$ and equality holds if α is not an integer. We could also use $f_{\alpha,q}^\sharp$ $(q \leqq p)$ in place of $f_\alpha^\sharp$ in the definition of C_p^α. However, in light of the inequalities (Theorem 4.3) $f_\alpha^\sharp \leqq f_{\alpha,q}^\sharp \leqq c\, M_\sigma(f_\alpha^\sharp)$ with $\sigma = (1/q + \alpha/n)^{-1}$ and the fact that M_σ is bounded on L_p, it follows that $f_{\alpha,q}^\sharp \in L_p$ is equivalent to $f_\alpha^\sharp \in L_p$ for $1 \leqq q \leqq p$. Also for $0 < q < 1$, we have $f_{\alpha,q}^\sharp \leqq f_\alpha^\sharp \leqq c\, M_{\sigma_0} f_{\alpha,q}^\sharp$ with $\sigma_0 : = (1 + \alpha/n)^{-1}$. Since M_{σ_0} is bounded on $L_p(\Omega)$, $f_{\alpha,q}^\sharp \in L_p(\Omega)$ is equivalent to $f_\alpha^\sharp \in L_p(\Omega)$ in this case as well. Similar statements hold for $f_\alpha^\flat$ and $f_{\alpha,q}^\flat$.

If $f \in C_p^\alpha$, we define the seminorm

$$|f|_{C_p^\alpha} : = ||f_\alpha^\sharp||_{L_p(\Omega)}$$

and the norm

$$||f||_{C_p^\alpha} : = ||f||_{L_p(\Omega)} + |f|_{C_p^\alpha} .$$

Similarly, $|f|_{\dot{C}_p^\alpha} : = ||f_\alpha^\flat||_{L_p(\Omega)}$ and $||f||_{\dot{C}_p^\alpha} : = ||f||_{L_p(\Omega)} + |f|_{\dot{C}_p^\alpha}$.

The triangle inequality for the two norms follows from the subadditivity of the $\sharp$ and $\flat$ maximal operators which is an immediate consequence of the definition (2.2). Another useful inequality which follows from the subadditivity is

(6.1) $$|f_\alpha^\sharp(x) - g_\alpha^\sharp(x)| \leqq (f - g)_\alpha^\sharp(x) \qquad x \in \Omega$$

which holds whenever $g_\alpha^\sharp(x)$ is finite.

Lemma 6.1. For $1 \leqq p \leqq \infty$ and $\alpha > 0$, C_p^α and $\dot{C}_p^\alpha$ are Banach spaces under their respective norms.

Proof. We prove that C_p^α is complete with the proof for $\dot{C}_p^\alpha$ following in much the same way. Suppose $\{f_m\}$ is Cauchy in C_p^α. Since L_p is complete there exists an $f \in L_p$ such that $f_m \to f$ in L_p. If Q is a cube in $\mathbb{R}^n$, then whenever $h_m \to h$ in L_p there must hold

$$\frac{1}{|Q|^{1+\alpha/n}} \int_Q |h - P_Q h| = \lim_{m\to\infty} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |h_m - P_Q h_m|$$
$$\leqq \underline{\lim}_{m\to\infty} (h_m)_\alpha^\#(x) \quad , \qquad x \in Q$$

since the operator P_Q is bounded on $L_1(Q)$. Taking a supremum over all cubes Q containing x gives

$$(6.2) \qquad h_\alpha^\# (x) \leqq \underline{\lim}_{m\to\infty} (h_m)_\alpha^\# (x) \qquad x \in \Omega.$$

Applying this inequality to the sequence $\{f_m\}$, taking p-th powers, and applying Fatou's lemma, we deduce $\|f_\alpha^\#\|_{L_p} \leqq (\int \underline{\lim}_{m\to\infty} |(f_m)_\alpha^\#|^p)^{1/p} \leqq \underline{\lim}_{m\to\infty} \|f_m\|_{C_p^\alpha}$ and so $f \in C_p^\alpha$. Similar reasoning shows that inequality (6.2) applied to the sequence $\{f_m - f_{m'}\}_{m=1}^\infty$ gives

$$\|(f - f_{m'})_\alpha^\#\|_{L_p} \leqq \underline{\lim}_{m\to\infty} \|(f_m - f_{m'})_\alpha^\#\|_{L_p} .$$

But the right hand side converges to zero as $m' \to \infty$ since $\{f_m\}$ is Cauchy in C_p^α. Since $f_m \to f$ in L_p has already been established, $f_m \to f$ in C_p^α. □

The following result of Calderón [5] shows that $\dot{C}_p^\alpha$ is the Sobolev space $W_p^\alpha(\Omega)$ when α is an integer and $p > 1$.

Theorem 6.2. (Calderón) If k is a nonnegative integer, then for each $1 < p \leqq \infty$, $\dot{C}_p^k(\Omega) = W_p^k(\Omega)$ with equivalent norms.

Proof. We have shown in Theorem 5.6 that for $\mathcal{D}f := \sum_{|\nu|=k} |D^\nu f|$,

$$c_2\, \mathcal{D}f \leqq f_k^\flat \leqq c_1\, M(\mathcal{D}f\chi_\Omega) \qquad \text{a.e. on } \Omega$$

with M the Hardy Littlewood maximal operator. Since M is bounded on $L_p(\Omega)$, $p > 1$, we have

$$c_2 \, ||f||_{L_p(\Omega)} \leqq ||f_k^b||_{L_p(\Omega)} \leqq c_1 \, ||f||_{L_p(\Omega)}$$

provided $p > 1$. □

The spaces C_∞^α and $\mathcal{C}_\infty^\alpha$ can also be described in terms of classical smoothness. The following theorem shows that $C_\infty^\alpha = B_\infty^{\alpha,\infty}$ (see §3 for the definition of Besov spaces) when Ω is $\mathbb{R}^n$ or a cube in $\mathbb{R}^n$. More general domains are discussed in §11.

<u>Theorem 6.3</u>. If $\Omega = \mathbb{R}^n$ or a cube in $\mathbb{R}^n$, then $C_\infty^\alpha = B_\infty^{\alpha,\infty}$ with equivalent norms.

<u>Proof</u>. If $f \in C_\infty^\alpha$, then Theorem 2.5 shows that for $k = [\alpha] + 1$

$$\omega_k(f,t)_\infty \leqq c \, ||f_\alpha^\#||_{L_\infty(\Omega)} \, t^\alpha , \qquad t > 0.$$

Hence $f \in B_\infty^{\alpha,\infty}$ and $||f||_{B_\infty^{\alpha,\infty}} \leqq c \, ||f||_{C_\infty^\alpha}$.

On the other hand, if $f \in B_\infty^{\alpha,\infty}$, then for each Q there is a polynomial π of degree less or equal $[\alpha]$ (Theorem 3.5) such that

$$||f - \pi||_{L_\infty(Q)} \leqq c \, |f|_{B_\infty^{\alpha,\infty}} |Q|^{\alpha/n} .$$

Hence,

$$\frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - \pi| \leqq c \, |f|_{B_\infty^{\alpha,\infty}} .$$

Taking a sup over $Q \ni x$ and using Lemma 2.1, we observe that

$$f_\alpha^\#(x) \leqq c \, |f|_{B_\infty^{\alpha,\infty}} \qquad x \in \Omega$$

and hence $||f||_{C_\infty^\alpha} \leqq c \, ||f||_{B_\infty^{\alpha,\infty}}$. □

When α is not an integer, the space $B_\infty^{\alpha,\infty}$ is the same as the Lipschitz space Lip α. Recall that there are several definitions of the space Lip α. The following theorem shows that these definitions are equivalent when Ω is $\mathbb{R}^n$ or a cube in $\mathbb{R}^n$.

Theorem 6.4. Let Ω be $\mathbb{R}^n$ or a cube in $\mathbb{R}^n$ and $\alpha > 0$. For f locally integrable, the following conditions are equivalent:

i) there exists $M_1 > 0$ and functions $\{f_\nu\}_{|\nu|<\alpha}$ such that $f_0 := f$ and for each

$|\nu| < \alpha$ and for almost every $x \in \Omega$

$$f_\nu(y) = \sum_{|\mu+\nu|<\alpha} f_{\mu+\nu}(x) \frac{(y-x)^\mu}{\mu!} + R_\nu(x,y)$$

with $|R_\nu(x,y)| \leq M_1 \, |y - x|^{\alpha-|\nu|}$ a.e. $y \in \Omega$,

ii) there exists $M_2 > 0$ such that for almost every $x \in \Omega$, there is a polynomial P_x of degree less than α with

$|f(y) - P_x(y)| \leq M_2 \, |x - y|^\alpha$ a.e. $y \in \Omega$,

iii) for k the smallest integer $\geq \alpha$, there is an $M_3 > 0$ such that

$|\Delta_h^k (f,x)| \leq M_3 \, |h|^\alpha$ a.e x, $x + kh \in \Omega$,

iv) $f_\alpha^\flat \in L_\infty(\Omega)$.

In addition, if in i), ii), or iii) M_f denotes the smallest M_i for the corresponding property, then M_f is a seminorm equivalent to $||f_\alpha^\flat||_{L_\infty(\Omega)}$.

Proof. If i) holds, then ii) holds with $P_x(y) := \sum_{|\nu|<\alpha} f_\nu(x)\frac{(y-x)^\nu}{\nu!}$ and $M_2 = M_1$. If ii) holds and x, $x + kh \in \Omega$, then $\Delta_h^k(P_x,x) = 0$ since $\deg(P_x) < k$. Hence

$$\begin{aligned}|\Delta_h^k(f,x)| = |\Delta_h^k(f - P_x, x)| &\leq 2^k \max_{0\leq j\leq k} |f(y_j) - P_x(y_j)| \\ &\leq k^\alpha \, 2^k \, M_2 \, |h|^\alpha\end{aligned}$$

with $y_j := x + jh$, $j = 0, \ldots, k$. Hence iii) holds with $M_3 = k^\alpha \, 2^k M_2$.

If iii) holds, then according to Theorem 3.4-5 for each cube $Q \subset \Omega$ there is a polynomial π of degree less than α such that

$$||f - \pi||_{L_\infty(Q)} \leq c \, M_3 |Q|^{\alpha/n}.$$

Hence, if $x \in Q$,

$$\frac{1}{|Q|^{1+\alpha/n}} \int_Q |f - \pi| \leq c \, M_3.$$

Taking a supremum over all such Q and using Lemma 2.1, we see that $\|f_\alpha\|^b_{L_\infty(\Omega)} \leq c\, M_3$.

Finally, if condition iv) holds, then define $f_\nu := D_\nu f$ with $D_\nu f$ the Peano derivative whose existence is guaranteed by Lemma 5.2. The Peano derivatives satisfy for almost every x, $D_\nu f(x) := \lim_{Q\downarrow\{x\}} D^\nu(P^b_Q f)(x)$, $|\nu| = k$. Fix $x \in \Omega$ for which this holds.

Since

$$D^\nu(P^b_Q f)(y) = \sum_{|\mu+\nu|<\alpha} D^{\mu+\nu}(P^b_Q f)(x)\, \frac{(y-x)^\mu}{\mu!},$$

if $y \in \Omega$ with $f^b_\alpha(y) < \infty$, then

$$\begin{aligned} |R_\nu(x,y)| &= |f_\nu(y) - \sum_{|\mu+\nu|<\alpha} f_{\mu+\nu}(x)\, \frac{(y-x)^\mu}{u!}| \\ &\leq |D_\nu f(y) - D^\nu(P_Q f)(y)| \\ &\quad + \sum_{|\mu+\nu|<\alpha} |D^{\mu+\nu}(P_Q f)(x) - D_{\mu+\nu} f(x)|\, \frac{|(y-x)^\mu|}{\mu!} \end{aligned} \tag{6.5}$$

where Q is chosen as the smallest cube with $x,y \in Q \subset \Omega$. Inequality (5.4)' shows that

$$|D_\nu f(y) - D^\nu(P_Q f)(y)| \leq c\, f^b_\alpha(y)\, |Q|^{\frac{\alpha-|\nu|}{n}}$$

and also

$$\begin{aligned} \sum_{|\mu+\nu|<\alpha} |D^{\mu+\nu}(P_Q f)(x) - D_{\mu+\nu} f(x)|\frac{|(y-x)^\mu|}{\mu!} &\leq c\, f^b_\alpha(x) \sum_{|\mu+\nu|<\alpha} |Q|^{\frac{\alpha-|\mu+\nu|}{n}} |Q|^{\frac{|\mu|}{n}} \\ &\leq c\, f^b_\alpha(x)\, |Q|^{\frac{\alpha-|\nu|}{n}}. \end{aligned}$$

Substituting these estimates into inequality (6.5) gives

$$\begin{aligned} |R_\nu(x,y)| &\leq c\, [f^b_\alpha(y) + f^b_\alpha(x)]\, |x-y|^{\alpha-|\nu|} \\ &\leq c\, \|f^b_\alpha\|_{L_\infty} |x-y|^{\alpha-|\nu|} \end{aligned}$$

as desired, since $|Q|^{1/n} \leq c\, |x-y|$. □

Condition i) of Theorem 6.4 is the usual definition of a function in Lip α for Ω closed and is for example the standard hypothesis in the

Whitney extension theorem (cf. [15, p. 176]). Condition ii) is the characterization of Lipschitz functions due to H. Whitney [20]. We choose to adopt i) as the definition of the space Lip α (= Lip(α,Ω)) and define

$$|f|_{\text{Lip }\alpha} := \inf \{M\colon M \text{ satisfies i) of Theorem 6.4}\}$$

and

$$\|f\|_{\text{Lip }\alpha} := \|f\|_{L_\infty} + |f|_{\text{Lip }\alpha} .$$

<u>Corollary 6.5</u>. If Ω is $\mathbb{R}^n$ or a cube in $\mathbb{R}^n$ and $\alpha > 0$, then C_∞^α = Lip α with equivalent norms.

<u>Lemma 6.6</u>. Let $0 < \beta \leqq \alpha$ and $1 \leqq p \leqq \infty$. Then, there is a constant c independent of f such that

$$\|f\|_{C_p^\beta} \leqq c\, \|f\|_{C_p^\alpha} . \tag{6.6}$$

<u>Proof</u>. First suppose $p > 1$ and P is the projection operator of degree $[\alpha]$. From Lemma 2.3, we have

$$f_\beta^\#(x) \leqq \sup_{Q\ni x} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f-P_Qf| \leqq [\sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f-P_Qf|]^{1-\theta}\ [f_\alpha^\#(x)]^\theta \tag{6.7}$$

with $\theta\colon = \beta/\alpha$. When $x \in Q$, inequality (2.3) shows that

$$\frac{1}{|Q|} \int_Q |f-P_Qf| \leqq c\, Mf(x)$$

with M the Hardy-Littlewood maximal operator. Using this together with (6.7) gives

$$f_\beta^\# \leqq c\ [Mf]^{1-\theta}\ [f_\alpha^\#]^\theta \leqq c(Mf+f_\alpha^\#).$$

Applying L_p norms and using the fact that M is bounded on L_p readily gives (6.6).

For $p = 1$, we use the techniques of §4 to circumvent the fact that M is not bounded on L_1. Let $q\colon = (1+\beta/n)^{-1}$ and P_Qf denote a polynomial of best L_q approximation to f from $\mathbb{P}_{[\alpha]}$ on Q. Take $\theta\colon = \beta/\alpha$ and argue as in (6.7) to find

$$f^{\#}_{\beta,q}(x) \leq [\sup_{Q\ni x} (\frac{1}{|Q|} \int_Q |f-P_Qf|^q)^{1/q}]^{1-\theta} [f^{\#}_{\alpha,q}(x)]^\theta$$

$$\leq c [M_q f(x)]^{1-\theta} [f^{\#}_{\alpha,q}(x)]^\theta$$

where we used definition (4.2) and the fact that $\int_Q |f-P_Qf|^q \leq \int_Q |f|^q$. Also we have used Lemma 2.4.

It follows that

$$f^{\#}_{\beta,q} \leq c (M_q f + f^{\#}_{\alpha,q}) \leq c (M_q f + f^{\#}_\alpha),$$

where we used the fact that $f^{\#}_{\alpha,q} \leq f^{\#}_\alpha$ for $q \leq 1$. Taking an L_1 norm shows that

(6.8) $$||f^{\#}_{\beta,q}||_{L_1} \leq c (||f||_{L_1} + ||f^{\#}_\alpha||_{L_1}) = c ||f||_{C^\alpha_1} .$$

Finally, recall from Theorem 4.3 that $f^{\#}_\beta \leq c M_\sigma(f^{\#}_{\beta,q})$ with $\sigma := (1+\beta/n)^{-1}$. Since M_σ is bounded on L_1, we have $||f^{\#}_\beta||_{L_1} \leq c ||f^{\#}_{\beta,q}||_{L_1}$. When this is used in (6.8), the inequality (6.6) follows. □

The next result is a "reduction theorem" for the spaces C^α_p and $\dot{C}^\alpha_p$.

<u>Theorem 6.7</u>. Suppose $\alpha > 0$, $1 \leq p \leq \infty$, and $k < \alpha$. The space $\dot{C}^\alpha_p$ is equal to the space of functions $f \in L_p$ which have weak derivatives $D^\nu f \in \dot{C}^{\alpha-k}_p$ ($|\nu| = k$) and

(6.9) $$c_1 |f|_{\dot{C}^\alpha_p} \leq \sum_{|\nu|=k} |D^\nu f|_{\dot{C}^{\alpha-k}_p} \leq c_2 |f|_{\dot{C}^\alpha_p} .$$

Similarly, C^α_p is equal to the space of functions $f \in L_p$ with weak derivatives $D^\nu f$ in $C^{\alpha-k}_p$ ($|\nu| = k$) and

(6.10) $$c_1 |f|_{C^\alpha_p} \leq \sum_{|\nu|=k} |D^\nu f|_{C^{\alpha-k}_p} \leq c_2 |f|_{C^\alpha_p} .$$

<u>Proof</u>. Suppose $f \in \dot{C}^\alpha_p$. Corollary 5.7 shows that the weak derivatives $D^\nu f$ exist and equal the Peano derivatives, $|\nu| = k$. Let $\sigma := (1 + \frac{\alpha-k}{n})^{-1}$ and choose q so that $\sigma < q < 1 \leq p$; then inequality (5.4)' shows that for any cube $Q \subset \Omega$ with $x_0 \in Q$, the polynomial $\pi := D^\nu P^b_Q$ is of degree less than $\alpha-k$ and satisfies

$$\frac{1}{|Q|^{\frac{\alpha-k}{n}}}\left(\frac{1}{|Q|}\int_Q |D^\nu f - \pi|^q\right)^{1/q} \leq c\left(\frac{1}{|Q|}\int_Q (f_\alpha^b)^q\right)^{1/q} \leq c\, M_q(f_\alpha^b)(x).$$

Taking a supremum over all cubes Q with $x \in Q \subset \Omega$ shows that

$$(D^\nu f)_{\alpha-k,q}(x) \leq c\, M_q(f_\alpha)(x). \tag{6.11}$$

Since M_q is bounded on L_p, this gives

$$||(D^\nu f)_{\alpha-k,q}||_{L_p} \leq c\, |f|_{C_p^\alpha}.$$

Now it follows from Lemma 2.1 and Theorem 4.3 that for $\alpha' = \alpha - k$

$$(D^\nu f)_{\alpha'}^b \leq c(D^\nu f)_{\alpha',1}^b \leq c\, M_\sigma[(D^\nu f)_{\alpha',q}^b],$$

so since M_σ is bounded on L_p, we have

$$||(D^\nu f)_{\alpha-k}^b||_{L_p} \leq c\, |f|_{C_p^\alpha}.$$

This gives the right hand inequality in (6.9).

The right hand inequality in (6.10) is proved in the same way. The existence of the weak derivatives $D^\nu f$, $|\nu| = k$ follows from Lemma 6.6, the fact that $C_p^\beta = C_p^\beta$ if β is not an integer, and Corollary 5.7.

To prove the left hand inequality in (6.9), suppose $f \in L_p$ and $D^\nu f \in C_p^{\alpha-k}$, $|\nu| = k$. From Theorem 3.6, it follows that for each cube Q

$$\inf_{\pi \in \mathbb{P}(\alpha)} \int_Q |f-\pi| \leq c\, |Q|^{k/n} \sum_{|\nu|=k} \inf_{\pi_\nu \in \mathbb{P}(\alpha)-k} \int_Q |D^\nu f-\pi_\nu|.$$

If we divide both sides by $|Q|^{1+\alpha/n}$, take a supremum over all Q containing x and use Lemma 2.1, we find

$$f_\alpha^b(x) \leq c \sum_{|\nu|=k} (D^\nu f)_{\alpha-k}^b(x). \tag{6.12}$$

Applying L_p norms to (6.12) gives the desired result.

The same argument used in proving (6.12) shows that

$$f_\alpha^\#(x) \leq c \sum_{|\nu|=k} (D^\nu f)_{\alpha-k}^\#(x). \tag{6.13}$$

Hence, the left hand inequality in (6.10) follows by taking L_p norms. □

Up to this point, we have not defined the space C_p^0, $1 \leq p \leq \infty$. The following theorem (see [2]) will motivate our definition.

<u>Theorem 6.8</u>. Suppose $1 < p < \infty$ and f satisfies $\lim_{N\to\infty} (Mf)^*(N) = 0$ where M is the Hardy-Littlewood maximal operator, then

(6.14) $$c_1 \|f\|_{L_p} \leq \|f_0^\#\|_{L_p} \leq c_2 \|f\|_{L_p}$$

with c_1, c_2 independent of f.

<u>Proof</u>. The inequality $\|f_0^\#\|_{L_p} \leq c_2 \|f\|_{L_p}$ follows immediately from the facts that $f_0^\# \leq 2\, Mf$ and that the Hardy-Littlewood maximal operator M is bounded on L_p. To obtain the remaining left hand inequality in (6.9), for each $s > 0$ we define $E: = E_s: = \{Mf > (Mf)^*(2s)\} \cup \{f_0^\# > (f_0^\#)^*(2s)\}$. Then E is open and $|E| \leq 4s$. Now select for each x a dyadic cube Q(x) containing x which has smallest diameter and satisfies $Q(x) \cap E^c \neq \phi$. Subdividing Q(x) into 2^n congruent dyadic subcubes, we let $\tilde{Q}(x)$ be one of those that contains x, then necessarily $\tilde{Q}(x) \subset E$ and

(6.15) $$|Q(x)| = 2^n|\tilde{Q}(x)| \leq 2^n|Q(x) \cap E|.$$

But dyadic cubes have the property that when any two have intersecting interiors, then one must contain the other; hence we may select from the countable collection $\{Q(x)\}_{x\in E}$ countably many maximal cubes $\{Q_j\}_{j=1}^\infty$ whose interiors are pairwise disjoint and so that

(6.16) $$E \subset \bigcup_j Q_j, \qquad Q_j \cap E^c \neq \phi \text{ (each j)}, \quad \sum_j |Q_j| \leq 2^n|E|.$$

The last inequality follows from summing inequality (6.15) over all j to get

$$\sum_j |Q_j| \leq 2^n \sum_j |Q_j \cap E| = 2^n\, |E|.$$

Next we decompose f into two functions: $g: = \sum_j (f-f_{Q_j})\chi_{Q_j}$ and $h: = f-g = \sum_j f_{Q_j}\, \chi_{Q_j} + f\, \chi_{E^c}$. Since M is weak type (1,1) and strong type (∞,∞), then

(6.17) $$(Mf)^*(s) \leq (Mg)^*(s) + \|Mh\|_{L_\infty} \leq \frac{c}{s} \|g\|_{L_1} + \|h\|_{L_\infty}.$$

But $Q_j \cap E^c \neq \phi$, so we observe that

(6.18) $$\frac{1}{|Q_j|} \int_{Q_j} |f-f_{Q_j}| \leq \inf_{u\in Q_j} f_0^\#(u) \leq (f_0^\#)^*(2s)$$

and

$$|f_{Q_j}| \leq \inf_{u \in Q_j} Mf(u) \leq (Mf)^*(2s).$$

Moreover, $|f\chi_{E^c}| \leq (Mf)\chi_{E^c} \leq (Mf)^*(2s)$, so

$$||h||_{L_\infty} \leq \max \{\sup_j |f_{Q_j}|, ||f\chi_{E^c}||_{L_\infty}\} \leq (Mf)^*(2s).$$

Estimating the L_1 norm of g we have from (6.18) and (6.16) that

$$||g||_{L_1} \leq \sum_j \int_{Q_j} |f-f_{Q_j}| \leq \sum_j |Q_j| \, (f_0^{\#})^*(2s)$$

$$\leq 2^n \, |E| (f_0^{\#})^*(2s) \leq c \, s(f_0^{\#})^*(2s).$$

Combining these with (6.17) we obtain

$$(6.19) \qquad (Mf)^*(s) \leq c \, (f_0^{\#})^*(2s) + (Mf)^*(2s) \quad , \quad 0 < s < \infty.$$

Let $N > t > 0$ be arbitrary but fixed real numbers, then integrating (6.19) from $t/2$ to N with weight $1/s$ we obtain

$$\int_{t/2}^N (Mf)^*(s) \frac{ds}{s} \leq c \int_{t/2}^N (f_0^{\#})^*(2s)\frac{ds}{s} + \int_{t/2}^N (Mf)^*(2s)\frac{ds}{s}$$

$$\leq c \int_t^\infty (f_0^{\#})^*(s)\frac{ds}{s} + \int_t^{2N} (Mf)^*(s)\frac{ds}{s}$$

by changing variables. Subtracting the integral $\int_t^N (Mf)^*(s)\frac{ds}{s}$ from both sides and using the fact that $(Mf)^*$ decreases we see

$$(Mf)^*(t) \leq c \int_{t/2}^t (Mf)^*(s)\frac{ds}{s} \leq c \, [\int_t^\infty (f_0^{\#})^*(s)\frac{ds}{s} + \int_N^{2N} (Mf)^*(s)\frac{ds}{s}]$$

$$\leq c \, [\int_t^\infty (f_0^{\#})^*(s)\frac{ds}{s} + (Mf)^*(N)].$$

By letting $N \to \infty$ and using the hypothesis that $(Mf)^*(N) \to 0$ we find that for $t > 0$,

$$(6.20) \qquad (Mf)^*(t) \leq c \int_t^\infty (f_0^{\#})^*(s)\frac{ds}{s} .$$

But now we may use the fact that $|f| \leq Mf$ a.e. and apply Hardy's inequality to the integral in (6.20) to obtain

$$||f||_{L_p} \leq ||Mf||_{L_p} \leq c \, ||f_0^{\#}||_{L_p}$$

as desired. □

For $1 \leqq p < \infty$ we define the space C_p^0 to be L_p and set $||f||_{C_p^0} := ||f||_p$. For $p = \infty$, we define $C_\infty^0 :=$ BMO and $||f||_{C_\infty^0} := ||f||_{BMO} = ||f_0^{\#}||_\infty$. In view Theorem 6.8, these are the natural definitions for $1 < p < \infty$. However, some explanation is needed for the case $p = 1$. As we explain in §12, the proper definition for $p = 1$ is $f_{0,q}^{\#} \in L_1$ for some $q < 1$, which is equivalent to $f \in L_1$ modulo constants. With this definition C_p^0, $1 \leqq p \leqq \infty$, forms an interpolation scale. On the other hand, the space obtained by requiring $f^{\#} \in L_1$ implies $Mf \in L_1(\mathrm{loc})$ and so f belongs to $L \log L$ locally. This space does not form an interpolation scale with the L_p spaces $1 < p < \infty$.

§7. Comparison With Besov Spaces

We intend to carry further the study of the relationships of C_p^α and $\dot{C}_p^\alpha$ to the classical smoothness spaces. We shall assume that $\Omega = \mathbb{R}^n$ throughout this section. Similar arguments work for cubes in $\mathbb{R}^n$. Other domains are discussed in §11.

We start with some approximation estimates. For $1 \leqq p < \infty$ define

$$E_r(f,\rho,x)_p \ := \inf_{\pi \in \mathbb{P}_{r-1}} (\int_{Q_\rho(x)} |f-\pi|^p)^{1/p}$$

where $Q_\rho(x)$ is the cube centered at x with side length ρ and set

$$E_r(f,\rho)_p \ := \ ||E_r(f,\rho,\cdot)||_{L_p}.$$

From Theorem 3.4, it follows that whenever $g \in W_p^r$ then

$$E_r(g,\rho,x)_p \leqq c\ \rho^r \sum_{|\mu|=r} ||D^\mu g||_{L_p(Q_\rho(x))}.$$

Hence integrating over $x \in \mathbb{R}^n$, we get by Fubini's theorem that

$$(7.1) \qquad E_r(g,\rho)_p \leqq c\ \rho^r \sum_{|\mu|=r} (\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |D^\mu g(y) \chi_{Q_\rho(x)}(y)|^p \, dy\, dx)^{1/p}$$

$$\leqq c\ \rho^{r+n/p}\ ||g||_{W_p^r} \ .$$

Similarly, when $f \in L_p$

$$(7.2) \qquad E_r(f,\rho)_p \leqq [\int_{\mathbb{R}^n} (\int_{Q_\rho(x)} |f|^p) dx]^{1/p} \leqq c\ \rho^{n/p}\ ||f||_{L_p}.$$

Since E is subadditive, (7.1) and (7.2) give

$$E_r(f,\rho) \leqq E_r(f-g,\rho) + E_r(g,\rho) \leqq c\ \rho^{n/p}\{||f-g||_{L_p} + \rho^r ||g||_{W_p^r}\}.$$

Taking an infinum over all such g gives

$$E_r(f,\rho)_p \leqq c\ \rho^{n/p}\ K_r(f,\rho^r)_p$$

where $K_r(f,t)_p \ := K(f,t;L_p,W_p^r)$, $t > 0$, is the K functional for interpolation between L_p and W_p^r. It is known [11] that $K_r(f,t^r)_p \leqq c\ \omega_r(f,t)_p$ for $t > 0$. Thus

$$(7.3) \qquad E_r(f,\rho)_p \leqq c\ \rho^{n/p}\ \omega_r(f,\rho)_p.$$

The same estimate holds when $p = \infty$ with C in place of L_∞.

We are now in a position to prove the following continuous embedding theorem:

Theorem 7.1. If $1 \leq p \leq \infty$ and $\alpha > 0$, then we have the embeddings:

$$B_p^{\alpha,p} \to C_p^{\alpha} \to B_p^{\alpha,\infty}. \tag{7.4}$$

Proof. For $r := [\alpha] + 1$ we have from Theorem 2.5,

$$\|\Delta_h^r(f,\cdot)\|_{L_p} \leq c\, |h|^{\alpha} \|f_\alpha^{\#}\|_{L_p}$$

which leads immediately to the right hand embedding in (7.4).

To prove the left hand embedding, let $F := f_{\alpha,p}^{\#}$ with $f_{\alpha,p}^{\#}$ as in §4. By the observation (2.14) on the equivalence of maximal operators

$$F(x)^p \leq c \sup_{\rho>0} \{E_r(f,\rho,x)_p/\rho^{\alpha+n/p}\}^p$$

$$\leq c \int_0^\infty \frac{E_r(f,\rho,x)_p^p}{\rho^{\alpha p+n}} \frac{d\rho}{\rho}$$

because $E_r(f,\rho,x)$ is increasing as a function of ρ. Thus from (7.3)

$$\int_{\mathbb{R}^n} |F|^p \leq c \int_0^\infty \frac{E_r(f,\rho)_p^p}{\rho^{\alpha p+n}} \frac{d\rho}{\rho} \leq c \int_0^\infty \left(\frac{\omega_r(f,\rho)_p}{\rho^{\alpha}}\right)^p \frac{d\rho}{\rho}$$

$$\leq c \left(|f|_{B_p^{\alpha,p}} \right)^p .$$

Now $f_\alpha^{\#} \leq c\, f_{\alpha,1}^{\#} \leq c\, f_{\alpha,p}^{\#}$ and hence

$$\|f_\alpha^{\#}\|_{L_p} \leq c\, |f|_{B_p^{\alpha,p}}$$

as desired. □

Next we show that the embeddings in Theorem 7.1 are best possible within the scale of Besov spaces. We begin with the lower embedding.

Lemma 7.2. If $1 \leq p < \infty$ and $\alpha > 0$, then there is an f which belongs to $B_p^{\alpha,q}$ for each $p < q \leq \infty$, but $f \notin C_p^{\alpha}$.

Proof. Consider first the case $0 < \alpha < 1$ and $n = 1$. By the embedding $B_p^{\alpha,q} \subset B_p^{\alpha,\infty}$ it is obvious that we may assume $q < \infty$. Set $\delta := (1 + \frac{1}{p} - \alpha)^{-1}$ and $a := 2^{-\delta} < 1$. Consider the "hat" function

$$\psi(x) := \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad . \tag{7.5}$$

We select disjoint intervals $I_j := [a_j, b_j]$ with $\frac{1}{2}(b_j - a_j) = h_j := a^j$. Since $\Sigma\, h_j < \infty$, we can choose the intervals so they are all contained in $[0,A]$ with $A < \infty$. Define

$$f_j(x) := j^{-1/p}\, 2^j\, h_j\, \psi((x-a_j)/h_j).$$

Then f_j is supported on I_j. Further define

$$f := \sum_1^\infty f_j,$$

then

$$||f||_{L_p}^p \leq \sum_1^\infty (j^{-1/p}\, 2^j\, h_j)^p\, h_j \leq \sum_1^\infty [a^{\alpha p}]^j < \infty$$

so that $f \in L_p$.

To see that $f \notin C_p^\alpha$ notice that if $x \in I_j$,

$$f_\alpha^\#(x) \geq \frac{1}{|I_j|^{1+\alpha}} \int_{I_j} |f - f_{I_j}| = j^{-1/p}\, 2^j\, h_j^{1-\alpha}/2^{2+\alpha} \ . \tag{7.6}$$

Hence

$$\int_{\mathbb{R}} [f_\alpha^\#]^p \geq c \sum_1^\infty [j^{-1/p}\, 2^j\, h_j^{1-\alpha}]^p h_j = c \sum_1^\infty j^{-1} = \infty.$$

To estimate the Besov norm we need to estimate $||\Delta_s f||_{L_p}$ for $0 < s < a$. Choose k so that $h_{k+1} \leq s < h_k$. Then with c depending at most on p and α, we have

$$\begin{aligned} ||\Delta_s f||_{L_p} &\leq \sum_1^k ||\Delta_s f_j||_{L_p} + 2 \sum_{k+1}^\infty ||f_j||_{L_p} \\ &\leq c\, [\sum_1^k j^{-1/p}\, 2^j\, s\, h_j^{1/p} + \sum_{k+1}^\infty j^{-1/p}\, 2^j\, h_j^{1+1/p}] \\ &\leq c\, [s \sum_1^k j^{-1/p} (2a^{1/p})^j + \sum_{k+1}^\infty j^{-1/p} (2a^{1+1/p})^j] \qquad (7.7) \\ &\leq c\, [s\, k^{-1/p}\, a^{k(\alpha-1)} + k^{-1/p}\, a^{k\alpha}] \\ &\leq c\, s^\alpha\, |\log s|^{-1/p} \end{aligned}$$

where we've used the fact that $2a^{1/p} > 1$ and $2a^{1+1/p} < 1$. Inequality (7.7) gives $\omega(f,t)_p \leqq c\, t^\alpha |\log t|^{-1/p}$ for $0 < t < a$ and so

$$\int_0^a \left(\frac{\omega(f,t)_p}{t^\alpha} \right)^q \frac{dt}{t} \leqq c \int_0^a |\log t|^{-q/p} \frac{dt}{t} < \infty.$$

Also $\omega(f,t)_p \leqq 2\, ||f||_{L_p}$, hence

$$\int_a^\infty \left(\frac{\omega(f,t)_p}{t^\alpha} \right)^q \frac{dt}{t} \leqq 2\, ||f||_{L_p}^q \int_a^\infty t^{-\alpha q - 1}\, dt < \infty .$$

Thus $f \in B_p^{\alpha,q}$ when $p < q \leqq \infty$.

In the case $\alpha = 1$ and $n = 1$, the construction given above is also valid but it is necessary to make two changes in the estimates. In (7.6) we use the fact that f_j is even on I_j and therefore its best L_1 approximation by a linear function on I_j is the constant $(f_j)_{I_j}$. Hence inequality (7.6) is still valid. In the estimate (7.7) we replace Δ_s by Δ_s^2. The second sum is estimated in the same way with 2 replaced by 4 in the first inequality. For the first sum, we have $||\Delta_s^2 f_j||_{L_p} \leqq c\, j^{-1/p} 2^j s^{1+1/p}$ and therefore the sum is smaller than $c\, k^{-1/p} 2^k s^{1+1/p} \leqq c\, s\, |\log s|^{-1/p}$. This shows as before that $f \in B_p^{1,q}$.

Now consider the case $n > 1$ and $0 < \alpha \leqq 1$. Define $F(x_1,\ldots,x_n) := f(x_1)\, \phi(x_1,\ldots,x_n)$ where ϕ is infinitely differentiable with compact support and $\phi \equiv 1$ on $[0,A]^n$. Clearly $F \in L_p(\mathbb{R}^n)$. To estimate $\Delta_s F$ write

$$\Delta_s(F,x) = \phi(x+s)\, \Delta_{s_1}(f,x_1) + f(x_1)\, \Delta_s(\phi,x).$$

Since ϕ is smooth with compact support, $\phi \equiv 1$ on $[0,A]^n$, this gives

$$\begin{aligned} ||\Delta_s F||_{L_p(\mathbb{R}^n)} &\leqq c\, [\, ||\Delta_{s_1} f||_{L_p(\mathbb{R})} + ||f||_{L_p(\mathbb{R})} s \,] \\ &\leqq c\, |s|^\alpha |\log |s||^{-1/p} \end{aligned}$$

because of inequality (7.7) with c depending at most on p, α, and n. Similarly, for $\alpha = 1$,

$$\begin{aligned} \Delta_s^2(F,x) &= \Delta_{s_1}^2(f,x_1)\phi(x+2s) + f(x_1+s_1)\, \Delta_s^2(\phi,x) \\ &\quad + (f(x_1+s_1) - f(x_1))(\phi(x+2s) - \phi(x)) \end{aligned}$$

from which it follows that

$$||\Delta_s^2 F||_{L_p(\mathbb{R}^n)} \leq c\ |s|\ |\log|s||^{-1/p} .$$

Thus $F \in B_p^{\alpha,q}$ for $q > p$. But for any cube $Q = J_1 \times \ldots \times J_n$ $[0,A]^n$, we have $F_Q = f_{J_1}$ and $F(x) = f(x_1)$, $x \in Q$. Hence for each x with $x_1 \in I_j$, (7.6) gives

$$F_\alpha^{\#}(x) \geq j^{-1/p}\ 2^j\ h_j^{1-\alpha}/2^{\alpha+2} \qquad x \in [0,A]^n,\ x_1 \in I_j \tag{7.8}$$

from which it follows that $F_\alpha^{\#} \in L_p(\mathbb{R}^n)$ as desired.

Finally, for $\alpha' = k + \alpha$ with $0 < \alpha \leq 1$, let f_k satisfy $(f_k)^{(k)} = f$ with f as above and set $F_k := f_k\phi$ with ϕ as above. Since ϕ has compact support $F_k \in L_p(\mathbb{R}^n)$. Using Leibnitz's rule of differentiation one finds that $D^\nu F_k \in B_p^{\alpha'-k,q}(\mathbb{R}^n)$ for all $|\nu| = k$. Thus using the reduction theorems for Besov spaces, $F_k \in B_p^{\alpha',q}(\mathbb{R}^n)$ for $q > p$. Since $D^{ke_1}F_k = (f_k)^{(k)} = f$ on $[0,A]^n$, it follows from (7.6) that $D^{ke_1}F_k \notin C_p^{\alpha'-k}$. Hence Theorem 6.7 shows that $F_k \notin C_p^{\alpha'}$. □

Lemma 7.3. If $\alpha > 0$, then there is an f such that for each $1 \leq p \leq \infty$ and $1 \leq q < \infty$, $f \in C_p^\alpha$ but $f \notin B_p^{\alpha,q}$.

Proof. Consider first the case $n = 1$ and $0 < \alpha < 1$. We shall construct a function f in Lip α with compact support such that for sufficiently many x and s

$$|f(x+s) - f(x)| \geq c\ s^\alpha.$$

This will in turn show that $t^{-\alpha}\omega(f,t)_p \geq c$ for sufficiently many t and consequently $||f||_{B_p^{\alpha,q}} = \infty$. On the other hand f will be in C_p^α for all $1 \leq p \leq \infty$.

Fix a such that $0 < a < \min(5^{1/(\alpha-1)}, 24^{-1/\alpha})$ and set $A := a^{\alpha-1}$ and $\gamma := a^\alpha$. Then $A \geq 5$ and $0 < \gamma < \frac{1}{24}$. Let $h_j := a^j$, $m_j := A^j$ $(j = 1,2,\ldots)$ and ψ be as in (7.5). The dilated functions $\psi_j(x) := m_j h_j \psi(x/h_j)$ have support on $[0,2h_j]$ and $|(\psi_j)'| = m_j$ a.e. on that interval. With $M_j = [\frac{1}{2h_j}] - 1$ (where the brackets here denotes the greatest integer), define

$$f_j(x) := \sum_{i=0}^{M_j} \psi_j(x-2ih_j).$$

Hence f_j is supported on $[0,1]$. Now define the function f by $f: = \sum_j f_j$. Since $||f_j||_{L_\infty} \leqq m_j h_j = \gamma^j$, it follows that f is a bounded continuous function.

First we check that $f \in \text{Lip}\ \alpha$. If $a/2 > s > 0$, choose k so that $h_{k+1} \leqq 2s < h_k$, then

$$||\Delta_s f_j||_{L_\infty} \leqq \begin{cases} m_j s & \text{if } j \leqq k \\ 2||f_j||_{L_\infty} & \text{if } j > k\ . \end{cases}$$

Hence,

$$||\Delta_s f||_{L_\infty} \leqq \sum_1^k m_j s + 2 \sum_{k+1}^{\infty} m_j h_j$$

$$\leqq \frac{A}{A-1} m_k s + 2 \frac{\gamma^{k+1}}{1-\gamma}$$

$$\leqq 2(a^{\alpha-1})^k s + 4(a^\alpha)^{k+1} \leqq 10\ s^\alpha.$$

Since f is bounded, this shows that $f_\alpha^\# \in L_\infty$ (cf. Theorem 6.3). Observe further that if dist $(x,[0,1]) = : \delta > 0$, then

$$f_\alpha^\#(x) \leqq \sup_{|I|\geqq\delta} \frac{1}{|I|^{\alpha+1}} \int_I |f| \leqq \frac{1}{\delta^{\alpha+1}} \int_0^1 |f| \leqq \frac{c}{\delta^{\alpha+1}}$$

which shows that $f_\alpha^\# \in L_p$ for all $1 \leqq p \leqq \infty$.

Next we show that $f \notin B_p^{\alpha,q}$ for any $1 \leqq q < \infty$, $1 \leqq p \leqq \infty$. Fix k and let s satisfy $\frac{1}{3} h_k \leqq s \leqq \frac{1}{2} h_k$. Define the set

$$E_s := \bigcup_{j=0}^{M_k} [2jh_k, 2jh_k + h_k/2]$$

then $|E_s| \geqq \frac{1}{8}$, $E_s \subset [0,1]$, and for $x \in E_s$

$$|\Delta_s f(x)| \geqq |\Delta_s f_k(x)| - |\sum_{j\neq k} \Delta_s f_j(x)|$$

$$\geqq m_k s - \{\sum_1^{k-1} m_j s + 2 \sum_{k+1}^{\infty} m_j h_j\}.$$

But $\sum_{j=1}^{k-1} m_j \leqq \frac{1}{4} m_k$ and $\sum_{k+1}^{\infty} m_j h_j \leqq \frac{1}{4} m_k s$, so

(7.9) $$|\Delta_s f(x)| \geqq \frac{1}{4} m_k s \qquad \text{if } x \in E_s;\ s \in [\tfrac{1}{3}h_k, \tfrac{1}{2}h_k].$$

On the other hand,

$$s^{\alpha-1} \leqq 3m_k$$

and so by (7.9)

(7.10) $$|\Delta_s f(x)| \geqq \frac{1}{12} s^\alpha \qquad \text{if } x \in E_s.$$

Taking L_p norms we see that

$$\omega(f,s)_p \geqq ||\Delta_s f||_{L_p} \geqq ||(\Delta_s f)\chi_{E_s}||_{L_p} \geqq \frac{s^\alpha}{96}$$

at least if $s \in [\frac{1}{3} h_k, \frac{1}{2} h_k]$, $k = 1,2,\dots$. But since ω is monotone, $\omega(f,t)_p \geqq c\, t^\alpha$ for all $0 < t \leqq 1$. Hence

$$\int_0^1 [t^{-\alpha}\, \omega(f,t)_p]^q \frac{dt}{t} = \infty.$$

The same ideas work for $\alpha = 1$, $n = 1$ with the following modifications. We now take $A = 24$ and $a = \frac{1}{24}$. Set

$$\psi(t) := \begin{cases} t^2 & 0 \leqq t \leqq 1 \\ 2-(t-2)^2 & 1 \leqq t \leqq 3 \\ (t-4)^2 & 3 \leqq t \leqq 4 \\ 0 & \text{otherwise} \end{cases}$$

then $\psi_j(t) := m_j h_j^2 \psi(t/h_j)$ is continuously differentiable and $|\psi_j''| \leqq 2\, m_j$ a.e. In the definition of f_j we take $M_j := [\frac{1}{4h_j}] - 1$ and $f_j(x) := \sum_{i=1}^{M_j} \psi_j(x-4ih_j)$, then

$$||\Delta_s^2 f_j||_{L_\infty} \leqq \min(2m_j s^2, 8m_j h_j^2).$$

Hence the same arguments as above show that $||\Delta_s^2 f||_{L_\infty} \leqq c\, s$ and $f_1^\# \in L_p$ for all $1 \leqq p \leqq \infty$. On the other hand, arguing in a similar manner as in (7.9-7.10) will give $\omega_2(f,t) \geqq c\, t$, $0 < t \leqq 1$ and hence $f \notin B_p^{1,q}$ for all $1 \leqq q < \infty$ as desired.

For the case $0 < \alpha \leqq 1$ and $n > 1$, let

$$F(x_1,\dots,x_n) := f(x_1)\phi(x_1,\dots,x_n)$$

where f is as above and $\phi \equiv 1$ on $[0,1]^n$, is infinitely differentiable, and is supported on $R := [-1,2]^n$, then for $s = (s_1,\dots,s_n)$ and $0 < \alpha < 1$

$$||\Delta_s F||_{L_\infty} \leqq ||f||_{L_\infty} ||\Delta_s \phi||_{L_\infty} + ||\phi||_{L_\infty} ||\Delta_{s_1} f||_{L_\infty} \leqq c\, s^\alpha.$$

This shows that $F_\alpha^\# \in L_\infty$. Similarly for $\alpha = 1$, $\|\Delta_s^2 F\|_{L_\infty} \leq c\, s$ and so $F_1^\# \in L_\infty$. Also if $\delta(x) := \mathrm{dist}(x,R)$, then $F_\alpha^\#(x) \leq c\, \delta(x)^{-\alpha-n}$. Consequently, $F_\alpha^\# \in L_p$ for all $1 \leq p \leq \infty$. Since $|\Delta_s(F,x)| = |\Delta_{s_1}(f,x_1)|$, $x, x+s \in [0,1]^n$, it follows from (7.10) that

$$|\Delta_s(F,x)| \geq |\Delta_{s_1}(f,x_1)| \geq \frac{1}{12} s_1^\alpha \quad \text{if } x_1 \in E_{s_1} \text{ and } s_1 \in [\tfrac{1}{3} h_k, \tfrac{1}{2} h_k]. \tag{7.11}$$

This gives

$$\omega(F,t)_p \geq c\, t^\alpha, \quad 0 < t < 1$$

and therefore $F \notin B_p^{\alpha,q}$ for any $1 \leq q < \infty$. A similar argument with second differences shows that this follows for $\alpha = 1$ as well.

Finally, if $\alpha' = k + \alpha$ with $0 < \alpha \leq 1$, let f_k be such that $(f_k)^{(k)} = f$ with f as above and let $F_k := f_k \phi$ with ϕ as above. Then it is readily seen that $F_k \in C_p^{\alpha'}$ for $1 \leq p \leq \infty$ by the reduction theorem for C_p^α spaces (Theorem 6.7). On the other hand $D^{ke_1} F_k = F$ on $[0,1]^n$, therefore (7.11) shows that $D^{ke_1} F_k \notin B_p^{\alpha,q}$ if $q < \infty$. The reduction theorem [3] for Besov spaces then shows that $F_k \notin B_p^{\alpha',q}$ if $q < \infty$. □

<u>Corollary 7.4</u>. If $\alpha > 0$ and $1 \leq p < \infty$, then the space C_p^α is neither a Besov space nor a potential space.

<u>Proof</u>. In view of the embeddings of Theorem 7.1, the only possibility for C_p^α to be a Besov space is for it to equal $B_p^{\alpha,q}$ for some q with $p \leq q \leq \infty$. However, Lemmas 7.2 and 7.3 show that this is not the case.

If C_p^α were a potential space, it would have to be $\mathcal{L}_p^\alpha$ (see Stein [15] for notation). On the other hand [15, p. 155] for $p \geq 2$, $\mathcal{L}_p^\alpha \subset B_p^{\alpha,p}$ which would contradict Lemma 7.3 if $C_p^\alpha = \mathcal{L}_p^\alpha$. For $p \leq 2$, $\mathcal{L}_p^\alpha \subset B_p^{\alpha,2}$ which again would contradict Lemma 7.3 if $C_p^\alpha = \mathcal{L}_p^\alpha$. □

We now want to go a little deeper into the relationship between C_p^α, $\dot{C}_p^\alpha$ and the potential spaces $\mathcal{L}_p^\alpha$. If $\alpha = k$ is an integer and $1 < p < \infty$, then as we have shown in Theorem 6.2, $\dot{C}_p^k = W_p^k$ and as is well known $\mathcal{L}_p^k = W_p^k$. Hence $\dot{C}_p^k = \mathcal{L}_p^k$. Our next theorem gives embeddings when α is non-integral.

Theorem 7.5. If $0 < \alpha$ and $1 < p < \infty$, we have the continuous embeddings

(7.12) $$\mathcal{L}_p^\alpha \to \dot{C}_p^\alpha \to C_p^\alpha.$$

Proof. The right most embedding in (7.12) is well known to us since $f_\alpha^\# \leq c\, f_\alpha^\flat$. As noted above the left embeddings hold for α an integer. We will now use the complex method of interpolation to derive the case of arbitrary α from the case α an integer.

Let m be an integer such that $m < \alpha < m + 1$. Consider the maximal function

(7.13) $$f_\alpha(x) := \sup_{\rho>0} \rho^{-n-\alpha} \int_{Q_\rho(x)} |f - P_{Q_\rho(x)} f|$$

with Q_ρ the cube with side length ρ and center x and P the projection of degree $m + 1$. It follows from (2.14) i) and Lemma 2.3 that

(7.14) $$f_\alpha \leq f_\alpha^\# \leq c_1 f_\alpha.$$

It is clear that the supremum in (7.13) can be taken over ρ rational. Let $\{A_k\}_1^\infty$ be a sequence of sets such that A_k contains k positive rationals, $A_k \subset A_{k+1}$, $k = 1,2,\ldots$ and $\bigcup_1^\infty A_k$ is the set of positive rationals.
Define

(7.15) $$F_k(x) := \max_{\rho\in A_k} \rho^{-n-\alpha} \int_{Q_\rho(x)} |f - P_{Q_\rho(x)} f|,$$

then $F_k \uparrow f_\alpha$ and hence $||F_k||_{L_p} \uparrow ||f_\alpha||_{L_p}$. It follows from (7.14) that we need only show that

(7.16) $$||F_k||_{L_p} \leq c\, ||f||_{\mathcal{L}_p^\alpha}, \quad k=1,2,\ldots .$$

Fix $f \in \mathcal{L}_p^\alpha$. Next fix k and let $A_k = \{\rho_1,\ldots,\rho_k\}$. Define

$$\rho(x) := \sum_1^k \rho_j \chi_{S_j}$$

where S_j is the set of x such that the max in (7.15) is taken on for $\rho=\rho_j$ but not for any ρ_i with $i < j$. Since for each j, $\int_{Q_{\rho_j}(x)} |f - P_{Q_{\rho_j}(x)} f|$ is continuous, the function ρ is simple.

Consider now the family of operators S_z, $0 \leq \mathrm{Re}\, z \leq 1$ defined by

$$S_z g(x) := \rho(x)^{-n-m-z} \int_{Q_{\rho(x)}(x)} (g(y)-P_{Q_{\rho(x)}(x)}g(y))\ \phi(x,y)dy$$

$$= \sum_{j=1}^{k} \rho_j^{-n-m-z}\ \chi_{S_j}(x) \int_{Q_{\rho_j}(x)} [g(y)-P_{Q_{\rho_j}(x)}g(y)]\ \phi(x,y)dy$$

with $\phi(x,y) := \text{sign}[f(y)-P_{Q_{\rho(x)}(x)}f(y)]$. Going further, let $\mathcal{J}_z$ be the Bessel potential operators of order z. Using the form of S_z and the fact that $\mathcal{J}_z$ is operator valued analytic in Re z > 0, it follows that

$$T_z := S_z \circ \mathcal{J}_{z+m}$$

is an analytic family in the sense of Stein [17, p. 205]. Now let us estimate $T_{i\eta}g$ for $g \in L_p$ and $\eta > 0$. From the definition of S_z we have $||S_{i\eta}h||_{L_p} \leqq c\ ||h_m^{\#}||_{L_p} \leqq c\ ||h_m^{b}||_{L_p}$ therefore,

$$(7.17)\qquad ||T_{i\eta}g||_{L_p} \leqq c\ ||(\mathcal{J}_{m+i\eta}g)_m^{b}||_{L_p} \leqq c\ ||\mathcal{J}_{m+i\eta}g||_{\mathcal{L}_p^m}$$
$$\leqq c\ ||\mathcal{J}_m g||_{\mathcal{L}_\rho^m}\ (|\eta|+1)^n \leqq c\ ||g||_{L_p}\ (|\eta|+1)^n.$$

Here, we used the facts that $\mathcal{J}_{m+i\eta} = \mathcal{J}_{i\eta} \circ \mathcal{J}_m$, $C_p^m = \mathcal{L}_p^m$, $||\mathcal{J}_{i\eta}|| \leqq c(|\eta|+1)^n$ from L_p to L_p, and $\mathcal{J}_m$ is an isometry from L_p to $\mathcal{L}_p^m$. Similarly, we have

$$(7.18)\qquad ||T_{1+i\eta}g||_{L_p} \leqq c\ ||g||_{L_p}\ (|\eta|+1)^n .$$

This shows that T_z satisfies the hypothesis of the Stein interpolation theorem for analytic families. Thus for any $g \in L_p$,

$$||T_{\alpha-m}g||_{L_p} \leqq c\ ||g||_{L_p} .$$

Now since $f \in \mathcal{L}_p^\alpha$, there is a $g \in L_p$ such that $\mathcal{J}_\alpha g = f$ and $||g||_{L_p} = ||f||_{\mathcal{L}_p^\alpha}$. Hence

$$||F_k||_{L_p} = ||T_{\alpha-m}g||_{L_p} \leqq c\ ||g||_{L_p} = c\ ||f||_{\mathcal{L}_p^\alpha}$$

which is (7.16). □

Our final result of this section compares C_1^k to W_1^k. Although the interpolation spaces for (C_1^k, C_∞^k) and $(W_1^k, W_\infty^k = C_\infty^k)$ coincide for the real method (see §8), C_1^k is properly contained in W_1^k.

Lemma 7.6. Suppose Ω is $\mathbb{R}^n$ or a cube in $\mathbb{R}^n$ and k is a positive integer, then there is a function $f \in W_1^k$ which does not belong to C_1^k. Consequently,

$$C_1^k(\Omega) \subsetneq W_1^k(\Omega) \qquad k = 1,2.... .$$

Proof. The containment follows from Theorem 5.6. We will construct f to have compact support within Ω and so $||f||_{W_1^k(\Omega)} = ||f||_{W_1^k(\mathbb{R}^n)}$. Hence by a change of scale we may assume that $\Omega = [-1,1]^n$.

Consider first the case k = 1 and n = 1. Let ψ be an even C^∞ function with $\psi \equiv 1$ on $[-\frac{1}{4},\frac{1}{4}]$, $||\psi||_\infty = 1$, and supp $\psi \subset [-\frac{1}{e},\frac{1}{e}]$. Then define f to be odd with

$$f(x): = \begin{cases} (\log 1/x)^{-1}\ \psi(x) , & x > 0 \\ 0 , & x = 0 . \end{cases} \tag{7.19}$$

Notice that f is a continuous function which increases on $[-\frac{1}{4},\frac{1}{4}]$. Moreover, $||f||_{L_\infty} \leqq 1$, f is supported in $[-\frac{1}{e},\frac{1}{e}]$, and

$$f'(x) = x^{-1}\ (\log x)^{-2}\ \psi(x) + (\log 1/x)^{-1}\ \psi'(x), \qquad x > 0.$$

Since f is an odd function,

$$||f'||_{L_1} \leqq 2(\int_0^{1/e} (\log x)^{-2}\ \frac{dx}{x} + ||\psi'||_{L_\infty}) < \infty$$

and so $f \in W_1^1(\Omega)$. On the other hand, for $0 < x < 1/12$ (see §5 for notation)

$$\begin{aligned} N_1^1(f,x) &\geq \sup_{\rho>0} \frac{1}{\rho^2} \int_{x-\rho}^{x+\rho} |f(u)-f(x)|du \\ &\geqq \frac{1}{4}x^{-2} \int_{-x}^{3x} |f(u)-f(x)|du \geqq \frac{1}{4}x^{-2} \int_{-x}^{x} [f(x)-f(u)]du \\ &= \frac{1}{4}x^{-2} \int_{-x}^{x} \int_u^x f'(t)dt\ du \end{aligned} \tag{7.20}$$

where we've used the fact that f is an odd increasing function on $[-\frac{1}{4},\frac{1}{4}]$. But now, by changing the order of integration we see that

$$\begin{aligned} N_1^1(f,x) &\geqq \frac{1}{4}x^{-2} \int_{-x}^{x} (x+t)\ f'(t)dt \geqq \frac{1}{4x} \int_0^x f'(t)dt \\ &= \frac{1}{4}\ \frac{f(x)}{x} = \frac{1}{4}x^{-1}\ (\log 1/x)^{-1} , \quad 0 < x < 1/12 . \end{aligned}$$

Hence, from Corollary 5.4 $f_1^\flat \notin L_1(\Omega)$.

In case n > 1, let

$$F(x_1,...,x_n): = f(x_1)\ \phi(x_1,...,x_n)$$

where f is as above and ϕ is an infinitely differentiable function with support in $[-1,1]^n$ and $\phi \equiv 1$ on $[-\frac{1}{4},\frac{1}{4}]^n$. Obviously, $||F||_{L_\infty} \leqq 1$ and

$$\nabla F(x_1,\ldots,x_n) = \phi(x_1,\ldots,x_n)\, f'(x_1)\, e_1 + f(x_1)\, \nabla\phi(x_1,\ldots,x_n).$$

Hence $F \in W_1^1(\Omega)$. However a simple computation shows that $\int_Q F_1^\flat = \infty$, where $Q = [-\frac{1}{4},\frac{1}{4}]^n$.

For $k > 1$, we let f_k satisfy $f_k^{(k-1)} = f$ and $F_k := f_k\phi$ with f and ϕ as above. Using Leibnitz's rule of differentiation we find that $F_k \in W_1^k(\Omega)$. On the other hand, $D_{e_1}^{k-1}F_k = f\phi$ on Q, hence $D_{e_1}^{k-1}F_k \notin \dot{C}_1^1(\Omega)$. It follows from the reduction theorem for $\dot{C}_p^\alpha$ spaces (Theorem 6.7) that $F_k \in \dot{C}_1^k(\Omega)$. □

Actually, our proof could be slightly modified to show that there are constants c_1 and c_2 such that

$$c_1\, Mf'(x) \leqq f_1^\flat(x) \leqq c_2\, Mf'(x) \qquad -\frac{1}{4} \leqq x \leqq \frac{1}{4}$$

if f is any odd function which is continuous, increasing, and concave on [0,1]. The right hand inequality is (5.7).

The embeddings of this section are summarized in Figure I. Spaces connected by line segments indicate that the lower space is embedded in the upper space.

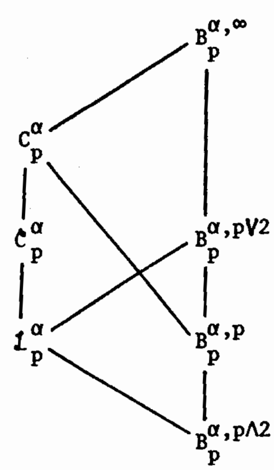

FIGURE I $(\alpha > 0;\ 1 < p < \infty)$

§8. Interpolation

We now examine some interpolation properties of the spaces C_p^α and $\dot{C}_p^\alpha$. It turns out that these spaces form interpolation scales for the real method of interpolation when α is fixed and p varies. We will show this by calculating the K functionals for the pairs $(C_1^\alpha, C_\infty^\alpha)$ and $(\dot{C}_1^\alpha, \dot{C}_\infty^\alpha)$. Recall that for any pair of Banach spaces (X_0, X_1) the K functional is defined for $f \in X_0 + X_1$ by

$$(8.1) \qquad K(f,t,X_0,X_1) := \inf_{f=f_0+f_1} \{||f_0||_{X_0} + t||f_1||_{X_1}\} \qquad t > 0 .$$

A key part of the calculation of these K-functionals is the Whitney extension theorem which extends a function f which is in Lip α on a closed set F to a function in Lip α on all of $\mathbb{R}^n$. We will need only a special case of this theorem for functions f which are defined on all of $\mathbb{R}^n$ to begin with. It will be convenient for us to give a formulation of the extension theorem for this special case in terms of the functions $f_\alpha^\#$ and $f_\alpha^\flat$.

Let f_α denote either of the functions $f_\alpha^\#$ or $f_\alpha^\flat$. Recall that the space $f_\alpha \in L_\infty(\mathbb{R}^n)$ is a Lipschitz space or generalized Lipschitz space (see §6). Suppose f is defined on $\mathbb{R}^n$ with $Mf \leq m_0$ and $f_\alpha \leq m_1$ on some closed set $F \subset \mathbb{R}^n$ where M is the Hardy-Littlewood maximal function. Then there is a function g such that g=f on F; $|g| \leq cm_0$ and $g_\alpha \leq cm_1$ on $\mathbb{R}^n$ with c a constant depending at most on n. Indeed, g can be constructed as follows.

Let $\{Q_j\}$ be a Whitney decomposition of F^c and ϕ_j^* the corresponding partition of unity (see [15, p. 167-170]). The Q_j have pairwise disjoint interiors, $\bigcup_j Q_j = F^c$ and for each j

$$(8.2) \qquad \operatorname{diam}(Q_j) \leq \operatorname{dist}(Q_j,F) \leq 4 \operatorname{diam}(Q_j).$$

The functions ϕ_j^* can be chosen to have support contained in cubes $Q_j^* := \frac{5}{4}Q_j$. Then

$$(8.3) \qquad \operatorname{diam}(Q_j^*) \leq c \operatorname{dist}(Q_j,F).$$

For each j, let $\tilde{Q}_j$ denote a cube with the same center as Q_j and side length $10\sqrt{n}$ times the side length of Q_j; then $\tilde{Q}_j \cap F \neq \emptyset$. The function g can then be defined as

$$g(x) := \begin{cases} f(x) & , \quad x \in F \\ \sum_j P_{\tilde{Q}_j} f(x)\, \phi_j^*(x) & , \quad x \in F^c \end{cases} \tag{8.4}$$

where P is the projection operator $P_{[\alpha]}$ (of degree $[\alpha]$) in case $f_\alpha = f_\alpha^\#$ and P is $P_{(\alpha)}$ (of degree (α)) in case $f_\alpha = f_\alpha^\flat$.

Lemma 8.1. If F is a closed set and f satisfies $Mf \leq m_0$ and $f_\alpha \leq m_1$ on F, then the function g defined by (8.4) satisfies:

i) $g = f$ on F; ii) $|g| \leq c\, m_0$ on $\mathbb{R}^n$; and iii) $g_\alpha \leq c\, m_1$ on $\mathbb{R}^n$.

Proof. From the definition of g, i) holds. To verify ii), first observe that $|g(x)| = |f(x)| \leq m_0$, $x \in F$. Now if $x \in F^c$, then since $\tilde{Q}_j \cap F \neq \emptyset$, it follows from (2.3) and our assumption that $Mf \leq m_0$ on F that $|P_{\tilde{Q}_j} f(x)| \leq c\, m_0$, $x \in \tilde{Q}_j$. Furthermore, supp $\phi_j^* \subset Q_j^* \subset \tilde{Q}_j$ and so $|P_{\tilde{Q}_j} f(x) \phi_j^*(x)| \leq c\, m_0\, \phi_j^*(x)$. Hence

$$|g(x)| \leq \sum_j c\, m_0\, \phi_j^*(x) = c\, m_0, \quad x \in F^c$$

since $\sum_j \phi_j^*(x) \equiv 1$, $x \in F^c$.

To verify iii), let Q be a cube in $\mathbb{R}^n$. We consider two cases: $Q \cap F \neq \emptyset$; $Q \subset F^c$. Consider first the case $Q \cap F \neq \emptyset$. Observe that if $Q \cap Q_j^* \neq \emptyset$, then $|Q| \geq c|Q_j|$ (because of (8.3)) and hence there is a cube R_j which contains both Q and $\tilde{Q}_j$ with $|R_j| \leq c|Q|$. It follows from (2.15) that

$$\|P_{\tilde{Q}_j} f - P_Q f\|_{L_\infty(Q \cap Q_j^*)} \leq \|P_{\tilde{Q}_j} f - P_{R_j} f\|_{L_\infty(\tilde{Q}_j)} + \|P_Q f - P_{R_j} f\|_{L_\infty(Q)}$$

$$\leq c[\inf_{u \in \tilde{Q}_j} f_\alpha(u)\, |R_j|^{\alpha/n} + \inf_{u \in Q} f_\alpha(u)\, |R_j|^{\alpha/n}] \tag{8.5}$$

$$\leq c\, m_1\, |Q|^{\alpha/n}$$

since both Q and $\tilde{Q}_j$ intersect F. Using (8.4), we can write $g - P_Q f = (f - P_Q f)\chi_F + \sum_j (P_{\tilde{Q}_j} f - P_Q f)\phi_j^*$. Hence from (8.5),

$$\begin{aligned} \int_Q |g-P_Q f| &\le \int_{Q\cap F} |f-P_Q f| + \sum_j \int_Q \|P_{\tilde{Q}_j} f-P_Q f\|_{L_\infty(Q\cap Q_j^*)}\, \phi_j^* \\ &\le \inf_{u\in Q\cap F} f_\alpha(u)\, |Q|^{1+\alpha/n} + c\, m_1\, |Q|^{\alpha/n} \int_Q (\sum_j \phi_j^*) \qquad (8.6) \\ &\le c\, m_1\, |Q|^{\alpha/n+1}. \end{aligned}$$

Now consider the second case $Q \subset F^c$. We have two possibilities: a) $|Q_{j_o}| > 4^n |Q|$ for some Q_{j_o} which intersects Q; b) $|Q_j| \le 4^n |Q|$ for all Q_j which intersect Q. In case a), we begin by showing that Q intersects at most $N^2 (N:=12^n)$ cubes Q_j^* and for each such j, $|Q_j| \le (4^n)^2 |Q_{j_o}|$. To see this we note that any neighbor of Q_{j_o} has measure $\ge 4^{-n} |Q_{j_o}| \ge |Q|$. Therefore Q is contained in the union of Q_{j_o} and its neighbors which number at most N. Now suppose $Q_j^* \cap Q \ne \phi$. Since Q_j^* is contained in the union of Q_j and its neighbors it follows that Q_j and Q_{j_o} have a common neighbor when $Q_j^* \cap Q \ne \phi$. But there are at most N^2 such Q_j and $|Q_j| \le (4^n)^2 |Q_{j_o}|$ as desired.

Let $k = [\alpha]$ or (α) according to whether f_α is $f_\alpha^\#$ or $f_\alpha^\flat$ and set $m: = k+1$. We estimate $D^\nu g$ for any $|\nu| = m$. By Leibnitz's formula, $D^\nu g = \sum_j \sum_{0<\mu\le\nu} \binom{\nu}{\mu} D^{\nu-\mu} P_{\tilde{Q}_j} f\, D^\mu \phi_j^*$. Note that $D^\nu P_{\tilde{Q}_j} f \equiv 0$, for each j and $\sum D^\mu \phi_j^* \equiv 0$ on F^c for $\mu > 0$. Thus we have

$$D^\nu g(x) = \sum_{Q_j^*\cap Q\ne\emptyset} \sum_{0<\mu\le\nu} \binom{\nu}{\mu} D^{\nu-\mu}(P_{\tilde{Q}_j} f - P_{\tilde{Q}_{j_o}} f)(x)\, D^\mu \phi_j^*(x). \qquad (8.7)$$

Using (2.15), the same argument as (8.5) shows that

$$\begin{aligned} \|D^{\nu-\mu}(P_{\tilde{Q}_j} f - P_{\tilde{Q}_{j_o}} f)\|_{L_\infty(Q_j^*)} &\le c\, m_1\, |\tilde{Q}_{j_o}|^{(\alpha-|\nu-\mu|)/n} \\ &\le c\, m_1\, |Q_{j_o}|^{(\alpha-|\nu|+|\mu|)/n}. \end{aligned}$$

Here we used the fact that all the Q_j^* which intersect Q have comparable size to Q_{j_o}. Also, the functions ϕ_j^* satisfy ([15, p. 174])

$$\|D^\mu \phi_j^*\|_\infty \leq c\,|Q_j|^{-|\mu|/n} \leq c\,|Q_{j_o}|^{-|\mu|/n}.$$

Using these last two estimates back in (8.7) gives

$$\|D^\nu g\|_{L_\infty(Q)} \leq c\, m_1 \sum_{Q_j^*\cap Q\neq\emptyset} |Q_{j_o}|^{(\alpha-m)/n} \leq c\, m_1\, |Q|^{(\alpha-m)/n}.$$

Hence from Theorem 3.4, there is a polynomial π of degree k such that

$$\|g-\pi\|_{L_\infty(Q)} \leq c\, m_1\, |Q|^{\alpha/n}.$$

Integrating gives

$$\frac{1}{|Q|^{1+\alpha/n}} \int_Q |g-\pi| \leq c\, m_1 . \tag{8.8}$$

Finally, we have case b). In this case, we can choose a cube $\tilde{Q}$ of measure $\leq c\,|Q|$ such that $\tilde{Q}$ contains each $\tilde{Q}_j$ for which Q_j^* intersects Q. Then, using (2.15), $\|P_{\tilde{Q}}f - P_{\tilde{Q}_j}f\|_{L_\infty(Q_j^*)} \leq c \inf_{u\in\tilde{Q}_j} f_\alpha(u)\, |\tilde{Q}|^{\alpha/n} \leq c\, m_1\, |Q|^{\alpha/n}$, and so

$$|(g-P_{\tilde{Q}}f)(x)| \leq \sum_{Q_j^*\cap Q\neq\emptyset} \|P_{\tilde{Q}_j}f - P_{\tilde{Q}}f\|_{L_\infty(Q_j^*)}\, \phi_j^*(x)$$

$$\leq c\, m_1\, |Q|^{\alpha/n} \sum_j \phi_j^*(x) \leq c\, m_1\, |Q|^{\alpha/n}, \quad x\in Q.$$

Integrating gives

$$\frac{1}{|Q|^{1+\alpha/n}} \int_Q |g-P_{\tilde{Q}}f| \leq c\, m_1 , \tag{8.9}$$

hence the three inequalities (8.6), (8.8) and (8.9) show that

$$g_\alpha(x) \leq c\, m_1$$

as desired. □

The following theorem characterizes the K-functional for the couples $(C_1^\alpha, C_\infty^\alpha)$ and $(\dot{C}_1^\alpha, \dot{C}_\infty^\alpha)$. The decomposition used below can be found in A. P. Calderón [5].

Theorem 8.2. If $\alpha > 0$, there exists constants $c_1, c_2 > 0$ such that

$$c_1 \int_0^t [f^*(s) + (f_\alpha^{\#})^*(s)]ds \leqq K(f,t;C_1^\alpha,C_\infty^\alpha) \tag{8.10}$$

$$\leqq c_2 \int_0^t [f^*(s) + (f_\alpha^{\#})^*(s)]ds, \quad t > 0$$

and

$$c_1 \int_0^t [f^*(s) + (f_\alpha^{b})^*(s)]ds \leqq K(f,t;\mathcal{C}_1^\alpha,\mathcal{C}_\infty^\alpha) \tag{8.11}$$

$$\leqq c_2 \int_0^t [f^*(s) + (f_\alpha^{b})^*(s)]ds, \quad t > 0.$$

Proof. We will only give the proof of (8.10). The proof of (8.11) is the same. First suppose $f = g + h$ with $g \in C_\infty^\alpha$ and $h \in C_1^\alpha$. Since $F \to F_\alpha^{\#}$ and $F \to F^{**}(t) := \frac{1}{t} \int_0^t F^*(s)ds$ are subadditive

$$\int_0^t [f^*(s) + (f_\alpha^{\#})^*(s)]ds \leqq \int_0^t [h^*(s) + h_\alpha^{\#*}(s)]ds + \int_0^t [g^*(s) + g_\alpha^{\#*}(s)]ds$$

$$\leqq \int_0^\infty (h^*(s) + h_\alpha^{\#*}(s))ds + t\,(||g||_\infty + ||g_\alpha^{\#}||_\infty)$$

$$= ||h||_{C_1^\alpha} + t||g||_{C_\infty^\alpha} .$$

Taking an infimum over such decompositions gives the left hand side of (8.10).

For the right hand inequality in (8.10), let $E := \{x: f_\alpha^{\#}(x) > (f_\alpha^{\#})^*(t)\} \cup \{x: Mf(x) > (Mf)^*(t)\}$ and $F := E^c$; then $|E| \leqq 2t$. If g is defined as in (8.4), then according to Lemma 8.1,

$$t\,||g||_{C_\infty^\alpha} = t\,(||g||_{L_\infty} + ||g_\alpha^{\#}||_{L_\infty}) \leqq c\,[t(Mf)^*(t) + tf_\alpha^{\#*}(t)] \tag{8.12}$$

$$\leqq c\,[\int_0^t f^*(s)ds + tf_\alpha^{\#*}(t)] \leqq c \int_0^t (f^*(s) + f_\alpha^{\#*}(s))ds$$

where we used the fact that $(Mf)^*(t) \leqq c\, f^{**}(t)$, $t > 0$, see [2].

We now want to estimate $h := f-g$ in the C_1^α norm. Let Q_j and $\tilde{Q}_j$ be as in the construction of g and define $\tilde{E} := \bigcup_j \tilde{Q}_j$ and $\tilde{F} = \tilde{E}^c$. Since $h \equiv 0$ on $F := E^c$, we have

$$||h||_{C_1^\alpha} = ||h||_{L_1} + ||h_\alpha^{\#}||_{L_1} = \int_E |h| + \int_{\tilde{E}} h_\alpha^{\#} + \int_{\tilde{F}} h_\alpha^{\#}. \tag{8.13}$$

The first two integrals are easy to estimate. Since $|E| \leqq 2t$,

$$(8.14)\qquad \int_E |h| \le \int_E |f| + |E|\ \|g\|_{L_\infty} \le c\,[\int_0^{2t} f^*(s)ds + t\,(Mf)^*(t)]$$

$$\le c\int_0^t f^*(s)ds$$

where we used the fact that $\int_0^{at} f^*(s)ds \le a\int_0^t f^*(s)ds$, $a \ge 1$. Similarly, using Lemma 8.1, we obtain

$$(8.15)\qquad \begin{aligned}\int_{\tilde E} h_\alpha^{\#} &\le \int_{\tilde E}(f_\alpha^{\#} + g_\alpha^{\#}) \le \int_0^{ct} f_\alpha^{\#*}(s)ds + |\tilde E|\ \|g_\alpha^{\#}\|_{L_\infty} \\ &\le c\,[\int_0^t f_\alpha^{\#*}(s)ds + t f_\alpha^{\#*}(t)] \le c\int_0^t f_\alpha^{\#*}(s)ds.\end{aligned}$$

In order to estimate the last integral in (8.13), we estimate $h_\alpha^{\#}$ on $\tilde F$. Suppose $x \in \tilde F$ and Q is a cube containing x. Then, since $h \equiv 0$ on $F \supset \tilde F$,

$$(8.16)\qquad \begin{aligned}\frac{1}{|Q|^{1+\alpha/n}}\int_Q h &\le \frac{1}{|Q|^{1+\alpha/n}}\sum_j \int_Q |f - P_{\tilde Q_j} f|\phi_j^* \\ &\le \sum_j \frac{1}{|Q|^{1+\alpha/n}} \int_{Q\cap Q_j^*} |f - P_{\tilde Q_j} f| .\end{aligned}$$

Now, $c\,|Q| \ge [\mathrm{dist}\,(x,Q_j)]^n$ whenever $Q \cap Q_j^* \ne \emptyset$ (recall $\mathrm{dist}(Q_j^*,F)$ is comparable to diam Q_j). Also, since $Q_j^* \subset \tilde Q_j$,

$$\int_{Q_j^*} |f - P_{\tilde Q_j} f| \le f_\alpha^{\#*}(t)\ |\tilde Q_j|^{1+\alpha/n} \le c\ f_\alpha^{\#*}(t)\ |Q_j|^{1+\alpha/n} .$$

Using this back in (8.16) and taking a sup over all such Q gives

$$(8.17)\qquad h_\alpha^{\#}(x) \le c\ f_\alpha^{\#*}(t) \sum_j \frac{|Q_j|^{1+\alpha/n}}{[\mathrm{dist}(x,Q_j)]^{\alpha+n}} \qquad x \in \tilde F.$$

Now, since $\mathrm{dist}(x,Q_j) \ge 2\,|Q_j|^{1/n}$ (recall the definition of $\tilde Q_j$)

$$\int_{\tilde F} [\mathrm{dist}(x,Q_j)]^{-\alpha-n}dx \le c\int_{2|Q_j|^{1/n}}^\infty \rho^{-\alpha-n}\rho^{n-1}d\rho \le c\ |Q_j|^{-\alpha/n}.$$

Hence integrating (8.17) gives

$$(8.18)\qquad \int_{\tilde F} h_\alpha^{\#} \le c\ f_\alpha^{\#*}(t)\sum_j |Q_j| \le c\ t\ f_\alpha^{\#*}(t) \le c\int_0^t f_\alpha^{\#*}(s)\ ds .$$

Therefore, the estimates (8.14), (8.15) and (8.18) used in (8.13) show that

$$\|h\|_{C_1^\alpha} \le c\int_0^t (f^*(s) + f_\alpha^{\#*}(s))ds.$$

This together with (8.12) proves the right hand estimate in (8.10). □

When X_1 and X_2 are Banach spaces with K functional $K(f,\cdot)$, and $0 < \theta < 1$; $0 < q \leqq \infty$, let $X_{\theta,q} := (X_1,X_2)_{\theta,q}$ denote the intermediate space (see [3,p. 167]) with

$$||f||_{X_{\theta,q}} := (\int_0^\infty [t^{-\theta}K(f,t)]^q \frac{dt}{t})^{1/q}$$

with the appropriate change when $q = \infty$. The spaces $X_{\theta,q}$ are interpolation spaces for (X_1,X_2). It follows from Theorem 8.2 and the Hardy inequality that $(C_1^\alpha, C_\infty^\alpha)_{1-1/p,p} = C_p^\alpha$ with equivalent norms. Similarly, $(\mathcal{C}_1^\alpha, \mathcal{C}_\infty^\alpha)_{1-1/p,p} = \mathcal{C}_p^\alpha$ with equivalent norms. Moreover, from the reiteration theorem for interpolation [3, p. 175], we have the following corollary.

Corollary 8.3. If $\alpha > 0$; $1 \leqq p \leqq q \leqq \infty$ and $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$ with $0 < \theta < 1$, then

i) $(C_p^\alpha, C_q^\alpha)_{\theta,r} = C_r^\alpha$ with equivalent norms,

ii) $(\mathcal{C}_p^\alpha, \mathcal{C}_q^\alpha)_{\theta,r} = \mathcal{C}_r^\alpha$ with equivalent norms.

As was pointed out to us by Peter Jones, it is also possible to use the decomposition of Theorem 8.2 to prove the interpolation theorem for Sobolev spaces (on $\mathbb{R}^n$) given by R. DeVore and K. Scherer [8]:

Theorem 8.4. If k is a positive integer, there exists constants $c_1, c_2 > 0$ depending at most on k and n such that for all $t > 0$

$$c_1 \int_0^t [f^*(s) + \sum_{|\nu|=k} (D^\nu f)^*(s)]ds \leqq K(f,t,W_1^k,W_\infty^k) \tag{8.19}$$
$$\leqq c_2 \int_0^t [f^*(s) + \sum_{|\nu|=k} (D^\nu f)^*(s)]ds .$$

Proof. The lower estimate follows in a simple way from the subadditivity of the map $F \to F^{**}$. For the upper estimate, as in the proof of Theorem 8.2, let $E := \{x: f_k^b(x) > f_k^{b*}(t)\} \cup \{x: Mf(x) > (Mf)^*(t)\}$ and take g as in (8.4) for $\alpha = k$ and $f_k := f_k^b$. Then using Theorem 6.2, and arguing as in (8.12),

$$||g||_{W_\infty^k} \leqq c||g||_{\mathcal{C}_\infty^k} \leqq c\ [\int_0^t f^*(s)ds + t\ f_k^{b*}(t)]. \tag{8.20}$$

It follows from Theorem 5.6 that $f_k^{b*}(t) \leqq c \sum_{|\nu|=k} (D^\nu f)^{**}(t)$ because $(MF)^* \leqq cF^{**}$ for any $F \in L_1 + L_\infty$. Hence (8.20) gives

$$t\, ||g||_{W_\infty^k} \leqq c \int_0^t (f^*(s) + \sum_{|\nu|=k} (D^\nu f)^*(s))ds. \tag{8.21}$$

Let $h := f-g$. Then $h \equiv 0$ on E^c and $|E| \leqq 2t$, so

$$\begin{aligned} ||h||_{L_1} &= \int_E |h| \leqq \int_E |f| + |E|\ ||g||_{L_\infty} \\ &\leqq c\,[\int_0^t f^*(s)ds + t\, f^{**}(t)] \leqq c \int_0^t f^*(s)ds. \end{aligned} \tag{8.22}$$

Also, using (8.21), we have for $|\mu| = k$,

$$\begin{aligned} ||D^\mu h||_{L_1} &\leqq \int_E |D^\mu h| \leqq \int_E |D^\mu f| + |E|\ ||D^\mu g||_{L_\infty} \\ &\leqq c \int_0^t [f^*(s) + \sum_{|\nu|=k} (D^\nu f)^*(s)]ds. \end{aligned} \tag{8.23}$$

Hence, (8.22) and (8.23) show that

$$||h||_{W_1^k} \leqq c \int_0^t [f^*(s) + \sum_{|\nu|=k} (D^\nu f)^*(s)]ds. \tag{8.24}$$

The inequalities (8.21) and (8.24) give the right hand inequality in (8.19). □

<u>Corollary 8.5</u>. If $1 \leqq p \leqq q \leqq \infty$ and $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$ with $0 < \theta < 1$, then

$$(W_p^k, W_q^k)_{\theta,r} = W_r^k \text{ with equivalent norms.} \tag{8.25}$$

Using the results of the previous section we show that the spaces C_p^α do not form an interpolation scale for the real method of interpolation if p is fixed.

<u>Theorem 8.6</u>. Suppose $1 \leqq p \leqq \infty$; $0 < \alpha_0 < \alpha_1$; $0 < \theta < 1$; and $1 \leqq r \leqq \infty$, then

$$(C_p^{\alpha_0}, C_p^{\alpha_1})_{\theta,r} = B_p^{\alpha,r} \tag{8.26}$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Consequently,

$$(C_p^{\alpha_0}, C_p^{\alpha_1})_{\theta,r} \neq C_q^\beta \tag{8.27}$$

for any values of $1 \leqq p < \infty$; $0 < \theta < 1$; $1 \leqq r \leqq \infty$; $1 \leqq q \leqq \infty$; $0 \leqq \beta$.

Proof. To prove (8.26) we see from Theorem 7.1 that

$$B_p^{\alpha_j,1} \to C_p^{\alpha_j} \to B_p^{\alpha_j,\infty}$$

and then apply the reiteration theorem [3, p. 175] for the real method of interpolation since

$$(L_p, W_p^k)_{\theta_j,1} = B_p^{\alpha_j,1}, \quad (L_p, W_p^k)_{\theta_j,\infty} = B_p^{\alpha_j,\infty} \qquad j = 0,1$$

where $k = [\alpha_1] + 1$ and $\theta_j = \alpha_j/k$, for example.

The fact (8.27) that the spaces C_p^α are not "stable" under the real method follows from (8.26) and Lemma's 7.2 and 7.3 which show that $B_p^{\alpha,r} \neq C_p^\alpha$ if $1 \leqq p < \infty$. □

§9. Embeddings

We shall now discuss Sobolev type embeddings for the spaces C_p^α. Embeddings for $\dot{C}_p^\alpha$ follow from these and the classical embeddings for Sobolev spaces. As a starting point, consider embeddings into the space C of continuous functions.

If R and R^* are cubes with $R^* \subset R$ and $|R| \leq 2^n |R^*|$, then (2.15) with $\nu = 0$ in gives

$$||P_R f - P_{R^*} f||_{L_\infty(R^*)} \leq c|R^*|^{\alpha/n} \inf_{u \in R^*} f_\alpha^\#(u) \leq c \int_{|R^*|/2}^{|R^*|} f_\alpha^{\#*}(s)\, s^{\alpha/n} \frac{ds}{s}.$$

More generally, given any two cubes $R^* \subset R$, choose $R_o \supset \ldots \supset R_m$ with $R_o := R$; $R_m := R^*$ and $2^n |R_j| = |R_{j-1}|$, $j=1,2,\ldots,m-1$; $|R_{m-1}| \leq 2^n |R_m|$. Then writing $P_{R^*} f - P_R f = \sum_1^m [P_{R_j} f - P_{R_{j-1}} f]$ gives

$$||P_R f - P_{R^*} f||_{L_\infty(R^*)} \leq c \int_{|R^*|/2}^{|R|} f_\alpha^{\#*}(s)\, s^{\alpha/n} \frac{ds}{s} . \tag{9.1}$$

If f is locally in L_1 on Ω, then according to (2.7) $\lim_{Q \downarrow \{x\}} P_Q f(x) = f(x)$, a.e. $x \in \Omega$. In view of (9.1), when $f_\alpha^\#$ is locally in the Lorentz space $L_{n/\alpha,1}$ (see [17, p. 188] for the definition) on Ω, then $\lim_{Q \downarrow \{x\}} P_Q f(x)$ exists for each $x \in \Omega$. Let $g(x) := \lim_{Q \downarrow \{x\}} P_Q f(x)$ so that $g(x) = f(x)$ a.e. Our next result shows that g is a continuous function and in turn gives an embedding of the space $\{f: f_\alpha^\# \in L_{n/\alpha,1}\}$ into C.

<u>Theorem 9.1</u>. If Ω is a domain and $f_\alpha^\#$ is locally in $L_{n/\alpha,1}$ on Ω, then there is a function $g \in C(\Omega)$ with $g = f$ a.e. on Ω. Moreover, if $f_\alpha^\# \in L_{n/\alpha,1}(\Omega)$ and Ω is $\mathbb{R}^n$ or a cube in $\mathbb{R}^n$, then there is a polynomial π of degree at most $[\alpha]$ such that

$$||g - \pi||_{C(\Omega)} \leq c\, ||f_\alpha^\#||_{L_{n/\alpha,1}(\Omega)} . \tag{9.2}$$

<u>Proof</u>. Let g be as above, then $g = f$ a.e. on Ω. We show that g is continuous. Let $R_o \subset \Omega$ be any cube and $u \in R_o$. If $Q \subset R_o$ is a cube, then choosing $R := Q$ and $R^* \downarrow \{u\}$ in (9.1) gives

(9.3) $$|P_Q f(u) - g(u)| \leqq c \int_0^{|Q|} F^*(s)\, s^{\alpha/n} \frac{ds}{s}$$

with $F: = f^{\#}_{\alpha,R_o}$ where the subscript R_o means that $f^{\#}_{\alpha}$ is defined as in (2.2) with R_o in place of Ω. Hence for any $x,y \in Q$

(9.4) $$|g(x) - g(y)| \leq c \int_0^{|Q|} F^*(s)\, s^{\alpha/n} \frac{ds}{s} + |P_Q f(x) - P_Q f(y)|.$$

Now $F(x) \leqq f^{\#}_{\alpha}(x)$, $x \in R_o$ and F is supported on R_o. Hence F is in $L_{n/\alpha,1}$. Thus, first choosing Q small, then fixing Q and letting $y \to x$ shows that g is continuous at x.

If $\Omega = R_o$ is a cube in $\mathbb{R}^n$, then (9.3) gives (9.2) with $\pi: = P_{R_o} f$. If $\Omega = \mathbb{R}^n$, take a sequence of cubes $\{Q_j\}_1^{\infty}$, with $Q_j \subset Q_{j+1}$ and $|Q_j| = 2^{jn}$, then using (9.1) we have for each $j < k$,

$$||P_{Q_j} f - P_{Q_k} f||_{C(Q_j)} \leqq c \int_{2^{j-1}}^{2^k} f^{\#*}_{\alpha}(s)\, s^{\alpha/n} \frac{ds}{s} \to 0 \quad \text{as } j,k \to \infty .$$

This shows that $\pi: = \lim_{j\to\infty} P_{Q_j} f$ exists and is a polynomial of degree at most $[\alpha]$ whenever $f^{\#}_{\alpha} \in L_{n/\alpha,1}(\mathbb{R}^n)$ and

$$||P_{Q_j} f - \pi||_{C(Q_j)} \leqq c \int_{2^{j-1}}^{\infty} f^{\#*}_{\alpha}(s)\, s^{\alpha/n} \frac{ds}{s} .$$

On the other hand, from (9.3)

$$||g - P_{Q_j} f||_{C(Q_j)} \leqq c \int_0^{2^j} f^{\#*}_{\alpha}(s)\, s^{\alpha/n} \frac{ds}{s}$$

and so

$$||g - \pi||_{C(Q_j)} \leqq c \int_0^{\infty} f^{\#*}_{\alpha}(s)\, s^{\alpha/n} \frac{ds}{s} = c||f^{\#}_{\alpha}||_{L_{n/\alpha,1}(\mathbb{R}^n)}.$$

Since j is arbitrary, this gives (9.2). □

The approach above can also be used to study classical differentiability of functions. We illustrate this by giving another proof of the following recent result of E. Stein [16].

<u>Theorem 9.2</u>. Let Ω be a domain in $\mathbb{R}^n$. If ∇f exists in the weak sense and is in $L_{n,1}(\Omega)$, then f can be redefined on a set of measure zero so as to be

continuous. Moreover, for this redefined f and for almost all $x \in \Omega$, $\nabla f(x)$ is the classical derivative of f: that is,

(9.5) $$|f(x+h) - f(x) - \nabla f(x)\cdot h| = o(|h|), \quad h \to 0.$$

Proof. We suppose that $n > 1$, since the case $n = 1$ is a classical result (Lebesgue's theorem for f') of real analysis due to the fact that $L_{1,1} = L_1$. Now, Theorem 5.6 and the boundedness of the Hardy Littlewood maximal operator M on $L_{n,1}$ show that the condition $|\nabla f| \in L_{n,1}$ implies $f_1^\flat \in L_{n,1}$. Since $f_1^\# \le c\, f_1^\flat$, Theorem 9.1 shows that f can be redefined on a set of measure zero so as to be continuous.

In order to prove (9.5), we can work locally and hence we assume for the remainder of the proof that Ω is a cube in $\mathbb{R}^n$ and f is continuous on Ω. Consider the maximal function

$$\Lambda f(x) := \overline{\lim_{h\to 0}} \frac{|f(x+h)-f(x)-h\cdot\nabla f(x)|}{|h|} .$$

We want to give a pointwise estimate between Λf and $T(f_1^\flat)$ where T is defined by

$$Tg(x) := \sup_{\Omega\supset Q\ni x} \frac{\|g\chi_Q\|_{L_{n,1}}}{\|\chi_Q\|_{L_{n,1}}} = \sup_{\Omega\supset Q\ni x} \frac{n}{|Q|^{1/n}} \|g\chi_Q\|_{L_{n,1}} .$$

Let $Q \subset \Omega$ be any cube. If $Q_2 \subset Q_1 \subset Q$ with $|Q_1| \le 2^n|Q_2|$, then

$$|f_{Q_1} - f_{Q_2}| \le \frac{c}{|Q_1|} \int_{Q_1} |f-f_{Q_1}| \le c \inf_{u\in Q_1} f_1^\flat(u)\, |Q_1|^{1/n}$$

$$\le c \int_{|Q_1|/2}^{|Q_1|} [f_1^\flat\chi_Q]^*(s)\, s^{1/n} \frac{ds}{s} .$$

The same telescoping argument as used in the derivation (9.3) shows that

$$|f(u) - f_Q| \le c \int_0^{|Q|} [f_1^\flat\chi_Q]^*(s)\, s^{1/n} \frac{ds}{s} = c\, \|f_1^\flat\chi_Q\|_{L_{n,1}} .$$

Hence, given x and h, we choose Q as a cube which contains x and x+h with $|Q| \le |h|^n$, and find

(9.6) $$|f(x+h) - f(x)| \le c\, \|f_1^\flat\chi_Q\|_{L_{n,1}} \le c\, T(f_1)(x)\, |h| .$$

From Theorem 5.6, we have $|\nabla f(x)| \le c\, f_1^\flat(x)\; c\, T(f_1^\flat)(x)$, a.e. $x \in \Omega$. Combining this with (9.6) shows that

(9.7) $$\Lambda f(x) \leqq c\, T(f_1^b)(x), \quad \text{a.e.} \quad x \in \Omega .$$

The sublinear operator T is easily seen to be of restricted weak type (n,n). Indeed,

$$T(\chi_E)(x) = \sup_{\Omega \supset Q \ni x} \frac{|E \cap Q|^{1/n}}{|Q|^{1/n}} = [M(\chi_E)(x)]^{1/n}$$

with M the Hardy-Littlewood maximal operator (for Ω). Recall that M is weak type (1,1). Since $n > 1$, restricted weak type implies weak type [17, p. 195] and so T is of weak type (n,n). In view of (9.7), there is a c such that

$$||\Lambda f||_{L_{n,\infty}(\Omega)} \leqq c\, ||f_1^b||_{L_{n,1}(\Omega)} .$$

Hence using Theorem 5.6,

(9.8) $$(\Lambda f)^*(t) \leqq c\, t^{-1/n}\, ||f_1^b||_{L_{n,1}(\Omega)} \leqq c\, t^{-1/n} |||\nabla f|||_{L_{n,1}(\Omega)} .$$

To complete the proof, note that $\Lambda(f-\phi) = \Lambda(f)$ when ϕ is smooth and so

$$(\Lambda f)^*(t) \leqq c\, t^{-1/n}\, |||\nabla(f-\phi)|||_{L_{n,1}(\Omega)} .$$

For any $\varepsilon > 0$, there is a smooth function ϕ with

$$|||\nabla(f-\phi)|||_{L_{n,1}(\Omega)} \leqq \varepsilon .$$

Therefore $(\Lambda f)^*(t) = 0$ for all t and so $\Lambda f = 0$ a.e. □

Remark: It is worth pointing out that f_1^b in (9.7) can be replaced by $|\nabla f|$ which can be proved directly (using Theorem 3.4) or deduced from (5.7).

To get embeddings of C_p^α into L_q or, more generally, C_q^β, we shall give an inequality between $f_\beta^\#$ and $f_\alpha^\#$ in terms of fractional integrals. Such an inequality for $\beta = 0$, $0 < \alpha < 1$ was given by A. P. Calderón and R. Scott [6] and we follow that idea in the general case. We assume for the remainder of this section that $\Omega = \mathbb{R}^n$ and $p \geqq 1$. More general domains are treated in §11 using extensions while the case $0 < p < 1$ is discussed in §12. Let P be the projection operator (2.1) of degree $[\alpha]$ and assume that $\beta < \alpha$ (and hence $[\beta] \leqq [\alpha]$). From Lemma 2.3, we have

(9.9)
$$f_\beta^{\#}(x) \leqq c \sup_{Q \ni x} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f-P_Q f|$$

whenever $f \in L_1 + L_\infty$. On the other hand for any cube $Q \ni x$ and any $0 < r < \frac{n}{\alpha-\beta}$, we have with $\gamma := r(\alpha-\beta) < n$,

(9.10)
$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |f-P_Q f| \leqq |Q|^{(\alpha-\beta)/n} \inf_{u \in Q} f_\alpha^{\#}(u) \leqq \{|Q|^{\gamma/n-1} \int_Q [f_\alpha^{\#}]^r\}^{1/r}$$
$$\leqq c \{\int_Q [f_\alpha^{\#}(y)]^r |x-y|^{\gamma-n} dy\}^{1/r}$$

because $|x-y| \leqq |Q|^{1/n}$ when $x,y \in Q$. Let I_γ denote the fractional integral operator

(9.11)
$$I_\gamma h(x) := \int_{\mathbb{R}^n} h(y) |x-y|^{\gamma-n} dy ,$$

then, returning to (9.9-10), we find

(9.12)
$$f_\beta^{\#}(x) \leqq c \{I_\gamma[(f_\alpha^{\#})^r](x)\}^{1/r} , \quad x \in \mathbb{R}^n .$$

Using (9.12) and the mapping properties of I_γ, we prove the following embeddings.

Theorem 9.3. Let $\Omega = \mathbb{R}^n$. If $0 \leqq \beta \leqq \alpha < \infty$, $1 \leqq p \leqq q < \infty$, and $\frac{1}{p} = \frac{1}{q} + \frac{\alpha-\beta}{n}$, then whenever $f \in L_1 + L_\infty$,

(9.13)
$$||f_\beta^{\#}||_{L_q} \leqq c ||f_\alpha^{\#}||_{L_p} .$$

Proof. The case $\beta = \alpha$ requires no proof, so suppose $\beta < \alpha$. The operator I_γ maps $L_{\tilde{p}}(\mathbb{R}^n)$ boundedly into $L_{\tilde{q}}(\mathbb{R}^n)$ whenever $1 < \tilde{p} < \tilde{q}$ and $1/\tilde{p} = 1/\tilde{q} + \gamma/n$ [15, p. 119]. Let $\tilde{p} := p/r$ and $\tilde{q} := q/r$ with $r < p$ and $r < n/(\alpha-\beta)$ as above. Then with $g := I_\gamma [(f_\alpha^{\#})^r]$, we have from (9.12)

$$||f_\beta^{\#}||_{L_q} \leqq c ||g^{1/r}||_{L_q} = c ||g||_{L_{\tilde{q}}}^{1/r} \leqq c ||(f_\alpha^{\#})^r||_{L_{\tilde{p}}}^{1/r} = c ||f_\alpha^{\#}||_{L_p} ,$$

which is (9.13). □

We concentrate now on the cases $q = \infty$ and $\beta = 0$.

Corollary 9.4. Let $\Omega = \mathbb{R}^n$, $1 \leqq p \leqq \infty$ and $\beta \geqq 0$. If $\alpha = \beta + n/p$ and $f \in L_1 + L_\infty$, then

(9.14) $$||f_\beta^{\#}||_{L_\infty} \leq c\ ||f_\alpha^{\#}||_{L_{p,\infty}}.$$

Proof. Starting with the left most inequality in (9.10), we have

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |f-P_Q f| \leq |Q|^{(\alpha-\beta)/n} \inf_{u\in Q} f_\alpha^{\#}(u)$$
$$\leq |Q|^{(\alpha-\beta)/n} f_\alpha^{\#*}(|Q|) \leq c\ ||f_\alpha^{\#}||_{L_{p,\infty}}.$$

Taking a supremum over all cubes Q proves (9.14). □

Recall the definition of the space C_p^o, that is $C_p^o: = L_p$, $1 \leq p < \infty$ and $C_\infty^o: = BMO$.

Corollary 9.5. Let $\Omega = \mathbb{R}^n$, $1 \leq p \leq q \leq \infty$ and $\alpha = n(\frac{1}{p} - \frac{1}{q})$. Then, there is a constant c independent of f such that

(9.15) $$||f||_{C_q^o} \leq c\ ||f||_{C_p^\alpha}.$$

Proof. For $q < \infty$, (9.13) gives

$$||f_o^{\#}||_{L_q} \leq c\ ||f_\alpha^{\#}||_{L_p}$$

when $f \in L_1 + L_\infty$. But $f \in C_p^\alpha$ implies $f \in L_p \subset L_1 + L_\infty$. Thus (9.15) holds when $q < \infty$. On the other hand when $q = \infty$, $f \in C_p^\alpha$ implies $f_\alpha^{\#} \in L_p \subset L_{p,\infty}$ and therefore (9.15) follows from (9.14). □

Our next result summarizes the embeddings of C_p^α into C_q^β. These are depicted in Fig. II where for fixed p and α, the shaded region indicates those pairs $(\frac{1}{q},\beta)$ for which $C_p^\alpha \to C_q^\beta$.

Theorem 9.6. Let $\Omega = \mathbb{R}^n$. If $1 \leq p \leq q \leq \infty$ and $0 \leq \beta \leq \alpha + n(\frac{1}{q} - \frac{1}{p})$, then

(9.16) $$C_p^\alpha \to C_q^\beta.$$

Proof. In view of Lemma 6.6, it is enough to consider the case $\beta = \alpha + n\ (\frac{1}{q} - \frac{1}{p})$. For this case we want to show $C_p^\alpha \to C_q^\beta$. There are two subcases depending on whether $\frac{1}{q_o} : = \frac{1}{p} - \frac{\alpha}{n}$ is non-negative or negative. In the first case, $C_p^\alpha \to C_{q_o}^o \cap L_p \to L_q$ because of Corollary 9.5 and Theorem 6.8. Also $||f_\beta^\#||_{L_q} \leq c\ ||f_\alpha^\#||_{L_p}$ because of Theorem 9.3. Hence (9.16) follows in this case.

Consider now the case $\frac{1}{p} - \frac{\alpha}{n}$ negative. Since $n/\alpha < p$ it follows that when $f_\alpha^\# \in L_p$ then $f_{\alpha,Q}^\# \in L_{n/\alpha,1}(Q)$ for each cube Q. Hence Theorem 9.1 gives that f can be redefined on a set of measure zero so as to be continuous and for each cube Q with $|Q| = 1$, the polynomial $P_Q f$ (since P is a projection onto $\mathbb{P}_{[\alpha]}$) satisfies

$$||f - P_Q f||_{C(Q)} \leq c\ ||f_\alpha^\#||_{L_{n/\alpha,1}(Q)} \leq c\ ||f_\alpha^\#||_{L_p} .$$

Inequality (2.3) implies that

$$||P_Q f||_{L_\infty(Q)} \leq c\ ||f||_{L_p(Q)} \leq c\ ||f||_{L_p} .$$

Hence

$$||f||_{C(Q)} \leq ||f - P_Q f||_{C(Q)} + ||P_Q f||_{C(Q)} \leq c\ ||f||_{C_p^\alpha} .$$

Since Q is arbitrary we have

$$||f||_C \leq c\ ||f||_{C_p^\alpha} .$$

This gives that $f \in C \cap L_p \subset L_q$.

To finish the proof, we note that when $q < \infty$ then (9.16) follows from Theorem 9.3 and when $q = \infty$, (9.16) follows from (9.14) and the fact that

$$||f_\alpha^\#||_{L_{p,\infty}} \leq ||f_\alpha^\#||_{L_p} \leq ||f||_{C_p^\alpha} . \quad \square$$

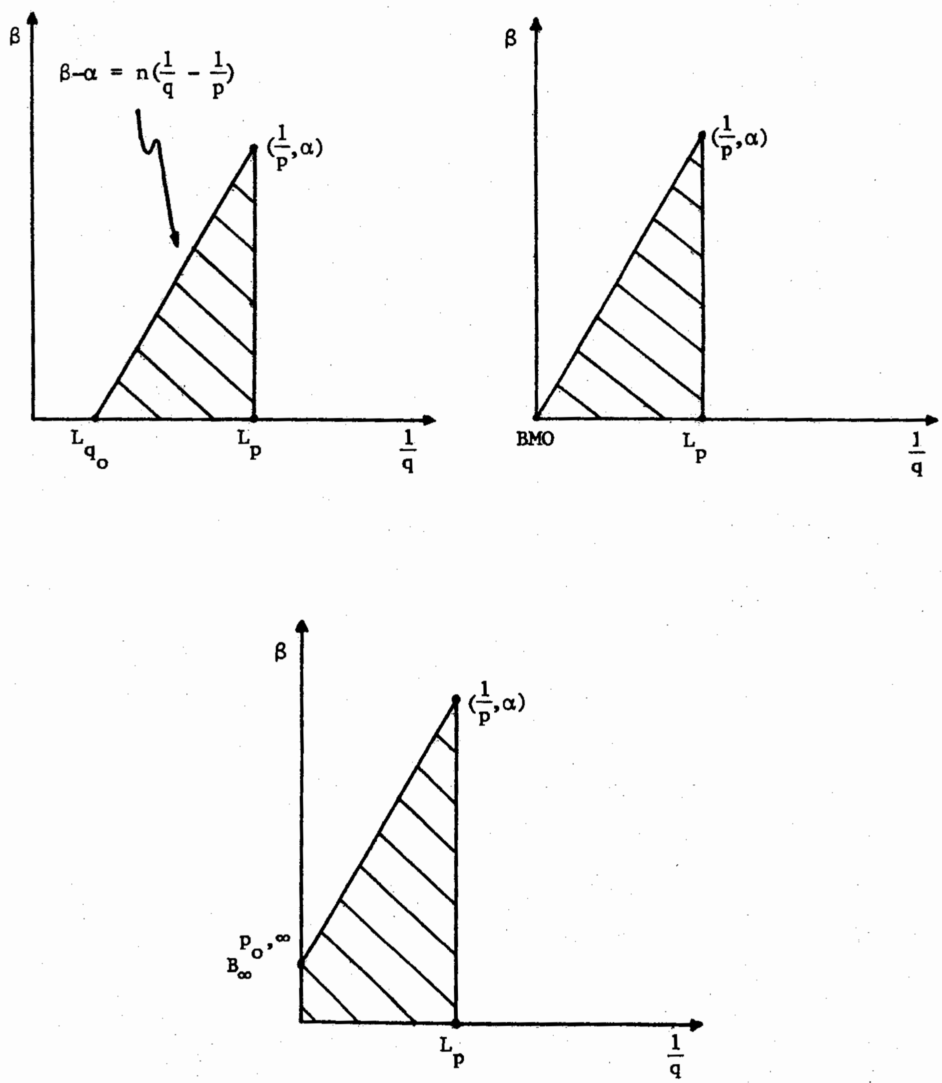

FIGURE II

Embeddings: $C_p^\alpha \to C_q^\beta$

§10. Extension Theorems

In the next section, we shall prove an extension theorems for the spaces $C_p^\alpha(\Omega)$ and $\dot{C}_p^\alpha(\Omega)$, $\alpha > 0$, $1 \leqq p \leqq \infty$ when Ω is a domain with a minimally smooth boundary in the sense of Stein [15, p. 189]. This will allow us to generalize various results of the previous sections (proved only for $\mathbb{R}^n$ or a cube in $\mathbb{R}^n$) to Ω. In the process, we show how the seminal ideas of Whitney [20] can be used to prove extension theorems for $1 \leqq p < \infty$. The original theorem of Whitney extends functions in Lip α on a closed set F to all of $\mathbb{R}^n$. Other extension theorems for Sobolev spaces W_p^k, $1 \leqq p < \infty$, are based on potentials as in the early work of Sobolev [14]. We should point out that most of the material in this section is obvious geometrically but rather detailed to prove analytically. The reader may benefit by convincing himself of the statements geometrically in lieu of the analytical arguments given.

We begin in this section by establishing extension theorems for domains $\Omega \subset \mathbb{R}^n$, $n > 1$ of the form $\Omega = \{(u,v)\colon\ u \in \mathbb{R}^{n-1}, v \in \mathbb{R} \text{ and } v > \phi(u)\}$ with ϕ a fixed function in Lip 1. That is, ϕ satisfies $|\phi(u_1) - \phi(u_2)| \leqq M|u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}^{n-1}$ and some M which we can take to be larger than 1. Later these extensions are pieced together to get the general case. The case $n = 1$ is discussed separately later in the section.

We need a decomposition of $(\partial\Omega)^c$ into dyadic cubes. In essence, we use the Whitney decompositions as described in [15, p. 167] with certain modification to meet our specific needs. As a starting point, note that the cone $C\colon = \{(u,v)\colon u \in \mathbb{R}^{n-1}, v \in \mathbb{R}; v > M|u|\}$ has the property that $x + C \subset \Omega$ whenever $x \in \Omega \cup \partial\Omega$ and $x - C \subset \Omega^c - \partial\Omega$ whenever $x \in \Omega^c \cup \partial\Omega$.

Let M_k, $k = 0, \pm 1, \ldots$ denote the collection of all dyadic cubes of side length 2^{-k} and $M\colon = \bigcup_{-\infty}^{\infty} M_k$. Each cube $Q \in M_k$ is contained in a cube $Q' \subset M_{k-1}$. We call Q' the parent of Q. For any cube Q and any $\tau > 0$ let τQ denote the cube with the same center as Q and side length $\tau\ell(Q)$ where $\ell(Q)$ is the side length of Q. Define F_o as the set of all cubes $Q \in M$ with center (u,v) such that either $4Q \subset (u,\phi(u)) + C$ or $4Q \subset (u,\phi(u)) - C$ (see Fig. III). Thus when $Q \in F_o$ then either $Q \subset \Omega$ or $Q \subset \Omega^c - \partial\Omega$. Further let F denote all the cubes $Q \in \Omega$ such that $Q \in F_o$ but the parent of Q is not in F_o. Similarly let F_c denote the set of all those cubes $Q \subset \Omega^c - \partial\Omega$ such that $Q \in F_o$ but the parent of Q is not in F_o.

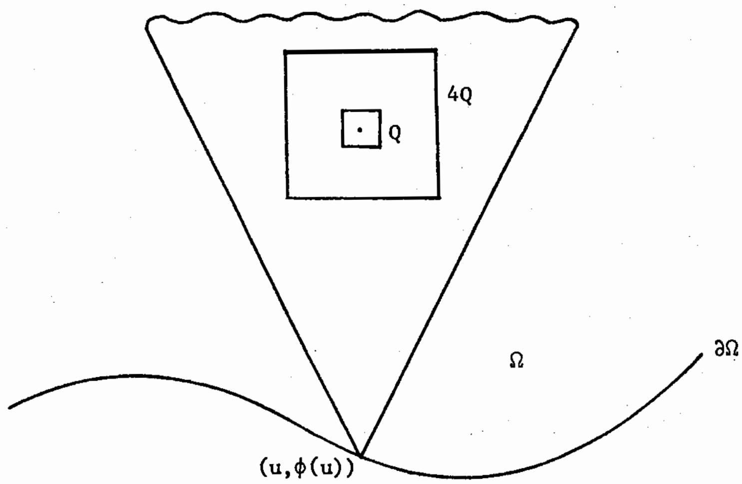

FIGURE III

Suppose now that $x = (u,v) \in \Omega^c\backslash\partial\Omega$, $(u \in \mathbb{R}^{n-1}, v \in \mathbb{R})$. Let x^s be the point in Ω which is "symmetric to x across $\partial\Omega$", i.e. $x^s := (u,\phi(u)+h)$ where $h = \phi(u)-v$. Our next lemma provides a procedure for reflecting cubes $Q \in F_c$ into cubes $Q^s \in F$.

Lemma 10.1. The cubes in F are a cover for Ω with pairwise disjoint interiors and the cubes of F_c are a cover for $\Omega^c - \partial\Omega$ with pairwise disjoint interiors. Also, there is a constant $c_o > 0$ depending only on n and M such that

(10.1) $\ell(Q) \leqq \operatorname{dist}(Q,\partial\Omega) \leqq c_o\, \ell(Q)$, $\quad Q \in F \cup F_c$

(10.2) $\sup_{(u,v)\in Q} |v - \phi(u)| \leqq c_o\, \ell(Q)$, $\quad Q \in F \cup F_c$

(10.3) For each Q in F_c, let Q^s be that cube in F which contains (u_o,v_o^s) where (u_o,v_o) is the center of Q; then

i) $c_o^{-1}\ell(Q) \leqq \ell(Q^s) \leqq c_o\, \ell(Q)$,

ii) $\operatorname{dist}(Q,Q^s) \leqq c_o\, \ell(Q)$,

iii) Each cube in F can be the symmetric cube Q^s of at most c_o cubes $Q \in F_c$.

Proof. First we make the observation that for $x: = (u,v) \in \Omega$, if Q is a dyadic cube containing x, then $Q \in F_o$ if Q is small enough (e.g., $\ell(Q) < (v-\phi(u))/(4+4M\sqrt{n})$). On the other hand if Q is too large (e.g., $\ell(Q) > v-\phi(u)$), then $Q \notin F_o$. Since dyadic cubes have the property that when any pair has intersecting interiors, one cube must be contained in the other, we have for each $x \in \Omega$ a maximal cube in F_o containing x. Since F is defined to be the collection of all such maximal cubes, then F is a cover for Ω whose members have pairwise disjoint interiors. The same argument shows that the cubes in F_c are a cover for Ω^c with pairwise disjoint interiors. If $Q \in F \cup F_c$, then $4Q \cap \partial\Omega = \phi$. Hence $\operatorname{dist}(Q,\partial\Omega) \geqq \frac{3}{2}\,\ell(Q) \geqq \ell(Q)$ which is the left hand inequality in (10.1). Suppose now that $Q \in F$ and Q' is the parent of Q. Since $Q' \notin F_o$ there is a point $(u',v') \in 4Q'$ with $v' \leqq \phi(u_o) + M|u'-u_o|$ where (u_o,v_o) is the center of Q'. Hence for any $(u,v) \in Q$

$$\begin{aligned} v-\phi(u) &\leqq v-v' + v'-\phi(u_o) + \phi(u_o)-\phi(u) \\ &\leqq 4\,\ell(Q') + M|u'-u_o| + M|u_o-u| \qquad (10.4) \\ &\leqq 4\,\ell(Q') + 4M\sqrt{n}\,\ell(Q') + M\sqrt{n}\,\ell(Q) \leqq A\ell(Q) \end{aligned}$$

with $A: = (9\,M\sqrt{n} + 8)$. A similar argument holds for $Q \in F_c$. This shows that (10.2) holds for any $c_o \geqq A$. Also, (10.2) implies the right hand side of (10.1).

Finally to see (10.3), let $Q \in F_c$. Since $Q^s \in F$, properties (10.1) and (10.2) imply

$$\ell(Q^s) \leqq \operatorname{dist}(Q^s,\partial\Omega) \leqq v_o^s - \phi(u_o) = \phi(u_o) - v_o \leqq c_o\, \ell(Q)$$

which verifies (10.3) i) if $c_o \geqq A$. The left hand inequality of i) follows similarly. By property (10.2) it is also clear that

$$\operatorname{dist}(Q,Q^s) \leqq c_o\, \ell(Q)$$

if $c_o \geqq 2A$ and so (10.3) ii) follows. Parts i) and ii) then show that iii) holds so long as $c_o \geqq A^2$. Hence if we define $\boxed{c_o := (9M\sqrt{n} + 8)^2}$, then all the conclusions of the lemma follow. □

Let us note some other properties of $F \cup F_c$. If Q_1, Q are two cubes in $F \cup F_c$ which touch, then according to (10.1),

$$\ell(Q_1) \leqq \operatorname{dist}(Q_1,\partial\Omega) \leqq \operatorname{dist}(Q,\partial\Omega) + \sqrt{n}\, \ell(Q) \leqq 2c_o\, \ell(Q) \tag{10.5}$$

so that Q_1 and Q have comparable size. It follows that there is a constant N depending only on n and M such that for each $Q_1 \in F \cup F_c$ at most N cubes Q from $F \cup F_c$ touch Q_1.

Now let $0 < \varepsilon \leqq c_o^{-1}$ and consider the cubes $\tilde{Q} := (1+\varepsilon)Q$ with $Q \in F \cup F_c$. We have the following property for the cubes $\tilde{Q}$:

(10.6) There is an N depending only on n and M such that each x appears in at most N of the cubes $\tilde{Q}$ with $Q \in F \cup F_c$.

Indeed, it follows from (10.5) that $\tilde{Q}$ is contained in the union of Q and all cubes in $F \cup F_c$ which touch Q. If $Q_1 \in F \cup F_c$ and $\tilde{Q}$ intersects Q_1, then Q_1 and Q must touch. As we observed above there are at most N such cubes. Hence (10.6) follows.

Now suppose $Q_1, Q \in F \cup F_c$ and $\operatorname{int}(\tilde{Q}_1) \cap \operatorname{int}(\tilde{Q}) \neq \phi$, then as we observed $\tilde{Q}$ is contained in the union of Q with all cubes in $F \cup F_c$ which touch Q. Similarly $\tilde{Q}_1$ is contained in the union of Q_1 and its neighbors. Therefore Q_1 and Q have a common neighbor and it follows from (10.5) that

(10.7) $\quad \ell(Q_1) \leq (2c_o)^2\, \ell(Q) \quad$ whenever $\text{int}(\tilde{Q}_1) \cap \text{int}(\tilde{Q}) \neq \phi$.

Let $Q_1, Q_2, \ldots$ be an enumeration of the cubes in F_c. Fix $\boxed{\varepsilon_o := (4c_o)^{-1}}$ and set $Q_j^* := (1+\varepsilon_o)Q_j$. Accordingly (see [15, p. 170]), there is a partition of unity $(\phi_j^*)_{j=1}^\infty$ with the properties:

$$\text{(10.8)} \quad \begin{array}{rl} \text{i)} & 0 \leq \phi_j^* \leq 1 \\ \text{ii)} & \Sigma\, \phi_j^* \equiv 1 \quad \text{on} \quad \Omega^c - \partial\Omega \\ \text{iii)} & \phi_j^* \text{ is supported in } \text{int}(Q_j^*) \\ \text{iv)} & \|D^\nu \phi_j^*\|_\infty \leq c\, [\ell(Q_j)]^{-|\nu|}. \end{array}$$

We can now define an extension operator E. Let $\alpha > 0$ be fixed and $P := P_{[\alpha]}$ be the projection in (2.1) of degree $[\alpha]$. If f is locally in $L_1(\Omega)$, define $E := E_\alpha^\#$ by

$$\text{(10.9)} \qquad Ef(x) := \begin{cases} f(x), & x \in \Omega \\ \sum_{k=1}^{\infty} P_{Q_k^s} f(x)\, \phi_k^*(x), & x \in \Omega^c - \partial\Omega\,. \end{cases}$$

We do not define Ef on the set $\partial\Omega$ which has measure 0. The extension operator $E_\alpha^\flat$ is defined in the same manner with $\mathbb{P}_{[\alpha]}$ now replaced by $\mathbb{P}_{(\alpha)}$ and so $E_\alpha^\# = E_\alpha^\flat$ if α is not an integer. In what follows, we will establish the mapping properties of $E_\alpha^\#$. The corresponding estimates for $E_\alpha^\flat$ simplify considerably and we will return this point later in the section.

We want now to estimate $(Ef)_\alpha^\#$. This requires us to estimate

$$\inf_{\pi \in \mathbb{P}_{[\alpha]}} \frac{1}{|R|^{1+\alpha/n}} \int_R |Ef-\pi|$$

for cubes R in $\mathbb{R}^n$. It turns out that the most difficult case is when R is close to the boundary of Ω and therefore we begin with this case.

If $Q \subset \Omega$ is cube in $\mathbb{R}^n$, then

$$\text{Shad}(Q) := \{(u,v) : v < \tilde{v},\ (u,\tilde{v}) \in Q\} \cap \Omega$$

is the <u>shadow</u> of Q (see Fig. IV).

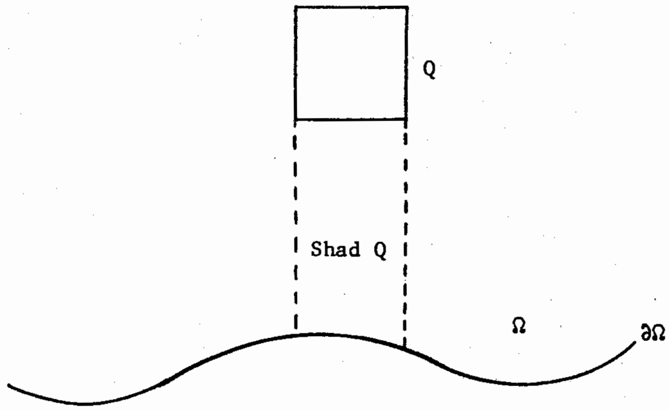

FIGURE IV

Shadow of Q

Lemma 10.2. There is a constant $c_1 > 0$ such that whenever $A \geqq 1$ and R is a cube in $\mathbb{R}^n$ with $\operatorname{dist}(R,\partial\Omega) \leqq A\,\ell(R)$, then there is a corresponding cube R_o with the following properties:

$$\text{(10.10)}\qquad \begin{array}{rl} \text{i)} & \ell(R_o) \leqq c_1\, A\, \ell(R), \\ \text{ii)} & 4\,R_o \subset \Omega, \\ \text{iii)} & v-\phi(u) \leqq c_1\, A\, \ell(R), \quad (u,v) \in R_o, \\ \text{iv)} & \text{if } Q \in F \text{ and } Q \cap R \neq \phi, \text{ then } Q \subset \operatorname{Shad}(R_o), \\ \text{v)} & \text{if } Q_j \in F_c \text{ and } Q_j^* \cap R \neq \phi, \text{ then } Q_j^s \subset \operatorname{Shad}(R_o). \end{array}$$

Proof. If $Q \in F \cup F_c$ and $Q \cap R \neq \phi$, then according to Lemma 10.1

$$\ell(Q) \leqq \operatorname{dist}(Q,\partial\Omega) \leqq \operatorname{dist}(R,\partial\Omega) + \sqrt{n}\,\ell(R) \leqq (A+\sqrt{n})\,\ell(R) \leqq 2\sqrt{n}\,A\,\ell(R)$$

and so $Q \subset (5\sqrt{n}\,A)R$. Similarly, if $Q_j \in F_c$ and $Q_j^* \cap R \neq \phi$, then a neighbor of Q_j, say $\tilde{Q} \in F_c$, intersects R. Hence (10.5) together with the last inequality shows that

$$(2c_o)^{-1}\,\ell(Q_j) \leqq \ell(\tilde{Q}) \leqq 2\sqrt{n}\,A\,\ell(R).$$

On the other hand, (10.3) ii) gives

$$\begin{aligned} \operatorname{dist}(R,Q_j^s) &\leqq \sqrt{n}\,\ell(Q_j^*) + \operatorname{dist}(Q_j,Q_j^s) \leqq (2c_o)\,\ell(Q_j) \\ &\leqq 8c_0^2\,\sqrt{n}\,A\,\ell(R). \end{aligned}$$

Now define $\boxed{\gamma := 24c_o^2\sqrt{n}}$ and $R_1 := \gamma A R$, then $Q, Q_j^s \subset R_1$ (the second containment use (10.3) i)) whenever Q and Q_j satisfy the assumptions in iv) and v). Next we observe for cubes $\tilde{R}_1 = \lambda e_n + R_1$ $(\lambda>0)$ that $Q, Q_j^s \subset \mathrm{Shad}(\tilde{R}_1)$ since $Q, Q_j^s \subset \Omega$. Define

$$R_o := c_o\, \ell(R_1)\, e_n + R_1.$$

Then R_o satisfies properties i), iv), and v) if $c_1 \geq \gamma$. Also one easily checks that $4R_o \subset (u_o, \phi(u_o)) + C \subset \Omega$ (where (u_o, v_o) is the center of R). Hence property ii) is also satisfied. Finally, we show inequality iii). If $(u,v) \in R_o$, we can find a $(u,v') \in R_1$ such that $v - v' = c_o\, \ell(R_1)$. Notice $R_1 \cap \partial\Omega \neq \phi$, so there is a point $(u_1, \phi(u_1)) \in R_1 \cap \partial\Omega$ and

$$\begin{aligned} v-\phi(u) &= v - v' + v' - \phi(u_1) + \phi(u_1) - \phi(u) \leq c_o\, \ell(R_1) + \ell(R_1) + M|u_1-u| \\ &\leq (c_o+1+M\sqrt{n})\, \ell(R_1) \leq c_1\, A\, \ell(R) \end{aligned}$$

where $\boxed{c_1 := (c_o+1+\sqrt{n}M)\gamma}$. Here we have used the inequality $|u_1-u| \leq \sqrt{n}\, \ell(R_1)$ in estimating $|\phi(u_1) - \phi(u)|$. Hence iii) holds. □

Let c_o be the constant of Lemma 10.1. Set $\boxed{A_o := 8c_o^2}$ and apply Lemma 10.2 with $A = A_o$ to obtain for each cube R, with $\mathrm{dist}(R,\partial\Omega) \leq A_o\, \ell(R)$, a cube R_o with the properties of Lemma 10.2. In particular, $\mathrm{dist}(R_o,\partial\Omega) \leq c_1\, A_o\, \ell(R_o)$ so Lemma 10.2 applies again to R_o with $A = c_1 A_o$. Let $\bar{R}$ be the cube guaranteed by Lemma 10.2 for R_o, then

$$(10.11)\qquad \begin{array}{ll} \text{i)} & \mathrm{dist}(\bar{R},\partial\Omega) \leq c_1^2\, A_o\, \ell(R) \\ \text{ii)} & \ell(\bar{R}) \leq c_1^2\, A_o\, \ell(R) \\ \text{iii)} & R_o \subset \mathrm{Shad}(\bar{R}) \\ \text{iv)} & Q \subset \mathrm{Shad}(\bar{R}) \ \text{ if } Q \cap R_o \neq \phi,\ Q \in \mathcal{F}\ . \end{array}$$

Although the cubes R_o and $\bar{R}$ are not uniquely determined by (10.10) and (10.11), the actual construction in Lemma 10.2 does produce a unique R_o. <u>For the remainder of this paper we take R_o and $\bar{R}$ to be unique cubes generated by the construction in Lemma 10.2</u>.

Lemma 10.3. Let R be a cube in $\mathbb{R}^n$ with $\text{dist}(R,\partial\Omega) \leq A_o\, \ell(R)$ and let R_o, R be the cubes described above; then

$$\int_R |Ef-P_{R_o} f| \leq c \int_{\text{Shad}(\bar{R})} f_\alpha^{\#}(y)\, \delta(y)^\alpha\, dy$$

where $\delta(y) := v-\phi(u)$ whenever $y = (u,v) \in \Omega$.

Proof. Let $Q \in F$ be any cube with $Q \subset \text{Shad } R_o$ and let (u_o,v_o) be its center. Choose a minimal number v_1 with $(u_o,v_1) \in R_o$. The line segment $\{(u_o,v)\colon v_o \leq v \leq v_1\}$ intersects a finite number of cubes from F as v ranges from v_1 down to v_o, say $R_1,R_2,\ldots,R_m = Q$. For each $j=2,\ldots,m$, R_j touches R_{j-1} and $\ell(R_j) \leq \ell(R_{j-1})$. Indeed, the translated cube $R_j' = \ell(R_j)\, e_n + R_j$ is a dyadic cube in F_o and intersects the interior of R_{j-1} nontrivially. Hence one of R_j' or R_{j-1} must contain the other. By the selection criteria for F, $R_j' \subset R_{j-1}$, so $\ell(R_j) \leq \ell(R_{j-1})$ and in fact

$$\text{Shad}(R_j) \subset \text{Shad}(R_{j-1}) \qquad j=2,3,\ldots,m. \tag{10.12}$$

We need the estimate

$$||P_Q f-P_{R_o} f||_{L_\infty(Q)} \leq c \sum_{j=0}^{m} m_{R_j} |R_j|^{\alpha/n} \tag{10.13}$$

where $m_{R_j} := \inf_{R_j} f_\alpha^{\#}$. To see this define $\tilde{R}_j := 4(R_{j-1})$, $2 \leq j \leq m$. Since $\ell(R_{j-1}) \geq \ell(R_j)$, it follows that $R_j \subset \tilde{R}_j$. For j=1, there is a common cube $\tilde{R}_1$ such that $\Omega \supset \tilde{R}_1 \supset R_1 \cup R_o$ and $\ell(\tilde{R}_1) \leq c\, \ell(R_1)$. Notice that $\tilde{R}_j \subset \Omega$ see (10.10) i)) and $Q \subset (2c_o+1)R_j$, $1 \leq j \leq m$, by the selection criteria for F and (10.2) respectively. Now using these facts, together with Lemma 3.2 and inequality (2.15), we see that

$$\begin{aligned}
||P_Q f - P_{R_o} f||_{L_\infty(Q)} &\leq \sum_{j=1}^{m} ||P_{R_j} f - P_{R_{j-1}} f||_{L_\infty((2c_o+1)\cdot R_j)} \\
&\leq c \sum_{j=1}^{m} ||P_{R_j} f - P_{R_{j-1}} f||_{L_\infty(R_j)} \\
&\leq c \sum_{j=1}^{m} [||P_{R_j} f - P_{\tilde{R}_j} f||_{L_\infty(R_j)} + ||P_{\tilde{R}_j} f - P_{R_{j-1}} f||_{L_\infty(R_j)}] \\
&\leq c \sum_{j=0}^{m} m_{R_j} |R_j|^{\alpha/n}
\end{aligned} \tag{10.14}$$

which verifies (10.13).

For such cubes Q we define the <u>tower</u> of Q by $T(Q) := \bigcup_{j=0}^{m} R_j$. Now it follows from (10.11) iv) that $T(Q) \subset \text{Shad}\,\bar{R}$ if $Q \cap R \neq \phi$, $Q \in F$. Hence,

$$\begin{aligned}
\int_Q |f - P_{R_o} f| &\leq \int_Q |f - P_Q f| + |Q|\, ||P_Q f - P_{R_o} f||_{L_\infty(Q)} \\
&\leq c\, |Q| \sum_{j=0}^{m} m_{R_j} |R_j|^{\alpha/n} \leq c\, |Q| \sum_{j=0}^{m} \int_{R_j} f_\alpha^{\#}(y)\, \delta(y)^{\alpha-n} dy \\
&= c\, |Q| \int_{T(Q)} f_\alpha^{\#}(y)\, \delta(y)^{\alpha-n} dy
\end{aligned} \tag{10.15}$$

since $|R_j|^{1/n}$ is comparable to $\delta(y)$ when $y \in R_j$ (see (10.2)for $j > 0$ and (10.10)iii) for $j = 0$) and $\{R_j\}_{j=1}^{m}$ are disjoint.

First we estimate the integral over $R \cap \Omega$; from (10.15)

$$\begin{aligned}
\int_{R\cap\Omega} |Ef - P_{R_o} f| &\leq \sum_{\substack{Q\in F \\ Q\cap R\neq\phi}} \int_Q |f - P_{R_o} f| \\
&\leq c \sum_{\substack{Q\in F \\ Q\cap R\neq\phi}} |Q| \int_{T(Q)} f_\alpha^{\#}(y)\, \delta(y)^{\alpha-n} dy \\
&= c \int_{\text{Shad}(\bar{R})} f_\alpha^{\#}(y)\, \delta(y)^{\alpha-n}\, \psi(y)\, dy \\
&\leq c \int_{\text{Shad}(\bar{R})} f_\alpha^{\#}(y)\, \delta(y)^{\alpha} dy
\end{aligned} \tag{10.16}$$

where $\psi(y) = \sum_{\substack{Q\in F \\ Q\cap R\neq\phi}} |Q|\, \chi_{T(Q)}(y)$. In the last inequality we use the fact that if $y = (u',v') \in T(Q)$, then either $y \in Q$ or Q is contained in the "cylinder" $\{(u,v): \phi(u) \leq v \leq v',\ |u-u'| \leq \sqrt{n}\, \delta(y)\}$. Hence, $\psi(y) \leq c\, \delta(y)^n$.

We can estimate $I := \int_{R\cap(\Omega^c\setminus\partial\Omega)} |Ef-P_{R_o} f|$ in much the same way. Namely, if $Q_j^* \cap R \neq \phi$, then Q_j^s is also a cube in F with $Q_j^s \subset \text{Shad}(R_o)$ and so the estimates used in (10.14-16) show that

$$\begin{aligned}(10.17)\quad &\sum_{Q_j^*\cap R\neq\phi} |Q_j^s|\ \|P_{Q_j^s}f-P_{R_o}f\|_{L_\infty(Q_j^s)} \\ &\leq c \sum_{Q_j^*\cap R\neq\phi} |Q_j^s| \int_{T(Q_j^s)} f_\alpha^{\#}(y)\ \delta(y)^{\alpha-n}dy \\ &\leq c \int_{\text{Shad}(\bar{R})} f_\alpha^{\#}(y)\ \delta(y)^{\alpha-n}\ \psi(y)dy \\ &\leq c \int_{\text{Shad}(\bar{R})} f_\alpha^{\#}(y)\ \delta(y)^{\alpha}dy\end{aligned}$$

since $Q_j^s \subset \text{Shad}(\bar{R})$. Here we used the fact that Q_j^s arises from at most c_o of the Q_j's because of (10.3) iii). Now since ϕ_j^* is supported on Q_j^* and $0 \leq \phi_j^* \leq 1$,

$$\begin{aligned}I &\leq \sum_{Q_j^*\cap R\neq\phi} \int_{Q_j^*} |P_{Q_j^s}f-P_{R_o}f|\phi_j^* \leq \sum_{Q_j^*\ R\neq\phi} |Q_j^*|\ \|P_{Q_j^s}f-P_{R_o}f\|_{L_\infty(Q_j^*)} \\ (10.18)\quad &\leq c \sum_{Q_j^*\cap R\neq\phi} |Q_j^s|\ \|P_{Q_j^s}f-P_{R_o}f\|_{L_\infty((c_o+1)^2Q_j^s)} \\ &\leq c \sum_{Q_j^*\cap R\neq\phi} |Q_j^s|\ \|P_{Q_j^s}-P_{R_o}f\|_{L_\infty(Q_j^s)}\end{aligned}$$

where we've used Lemma 3.2 and the facts that $|Q_j^*| \leq c_o^n\ (1+\varepsilon_o)^n|Q_j^s|$; $Q_j^* \subset (c_o+1)^2Q_j^s$ (by Lemma 10.1). The combination of (10.17-18) gives $I \leq c \int_{\text{Shad}(\bar{R})} f_\alpha^{\#}(y)\ \delta(y)^\alpha\ dy$, which together with (10.16) proves the Lemma. □

Define $\mathcal{Q} := \{Q: \text{dist}(Q,\partial\Omega) \leq A_o\ \ell(Q)\}$ and

$$\mu(f,x) = \sup_{\substack{Q\in\mathcal{Q}\\ Q\ni x}} \frac{1}{|Q|^{1+\alpha/n}} \int_{\text{Shad}(\bar{Q})} f_\alpha^{\#}(y)\ \delta(y)^\alpha\ dy$$

where $\bar{Q}$ is given according to (10.11). The following theorem gives the main estimate of this section.

<u>Theorem 10.4</u>. If f is locally in $L_1(\Omega)$, then

$$(Ef)_\alpha^{\#}(x) \leq c\ \mu(f,x) + f_\alpha^{\#}(x)\cdot\chi_\Omega(x), \qquad x \in \mathbb{R}^n.$$

Proof. Let R be an cube in $\mathbb{R}^n$. If $\operatorname{dist}(R,\partial\Omega) \leqq A_o\, \ell(R)$, then it follows from Lemma 10.3 that for $x \in R$

$$(10.19)\qquad \frac{1}{|R|^{\alpha/n+1}} \int_R |Ef-P_{R_o} f| \leqq \frac{c}{|R|^{\alpha/n+1}} \int_{\mathrm{Shad}(\bar{R})} f_\alpha^{\#}(y)\, \delta(y)^\alpha\, dy \leqq c\, \mu(f,x).$$

If $\operatorname{dist}(R,\partial\Omega) > A_o\, \ell(R)$, there are two cases depending on whether $R \subset \Omega$ or $R \subset \Omega^c - \partial\Omega$. In the first case, since $Ef = f$ on R, then for each $x \in R$,

$$(10.20)\qquad \frac{1}{|R|^{\alpha/n+1}} \int_R |Ef-P_R f| \leqq f_\alpha^{\#}(x)\, \chi_\Omega(x).$$

Consider now the second case $R \subset \Omega^c - \partial\Omega$ and $\operatorname{dist}(R,\partial\Omega) > A_o\, \ell(R)$. We first count how many of the cubes Q_j^* touch R. Let J be the set of all j such that $Q_j \in \mathcal{F}_c$ and $Q_j^* \cap R \neq \phi$, then, for $j \in J$,

$$\ell(R) \leqq \frac{1}{A_o} \operatorname{dist}(R,\partial\Omega) \leqq \frac{1}{A_o} [\operatorname{dist}(Q_j^*,\partial\Omega) + \sqrt{n}\, \ell(Q_j^*)]$$

$$\leqq \frac{1}{A_o} [\operatorname{dist}(Q_j,\Omega) + \frac{9\sqrt{n}}{8} \ell(Q_j)]$$

$$\leqq \frac{2c_o}{A_o} \ell(Q_j) \leqq (4c_o)^{-1} \ell(Q_j).$$

Hence, the cube $(1+(2c_o)^{-1})Q_j$ contains R. According to (10.6), there are at most N such cubes with N depending only on M and n; that is, $|J| \leqq N$.

Now take the largest cube Q_{j_o} with $j_o \in J$. For any other $j \in J$, $|Q_j| \geqq c\, |Q_{j_o}|$ because of (10.7). Also, $(1+(c_o)^{-1})Q_{j_o} \cap Q_j^* \neq \phi$ and hence $Q_j^* \subset 4Q_{j_o} =: \tilde{Q}$. We can use Lemma 10.2 for $\tilde{Q}$ because

$$\operatorname{dist}(\tilde{Q},\partial\Omega) \leqq \operatorname{dist}(Q_{j_o},\partial\Omega) \leqq c_o\, \ell(Q_{j_o}) \leqq c_o\, \ell(\tilde{Q}) < A_o\, \ell(\tilde{Q})$$

with c_o the constant of Lemma 10.1. Let $\tilde{Q}_o$ be the cube (for $\tilde{Q}$) guaranteed by Lemma 10.2. If $j \in J$, then $Q_j \subset Q_j^* \subset \tilde{Q}$ and $T(Q_j^s) \subset \mathrm{Shad}(\tilde{Q})^-)$, therefore the estimates in Lemma 10.3 show that for $x \in R \subset \tilde{Q}$,

$$||P_{Q_j^s} f - P_{\tilde{Q}_o} f||_{L_\infty(Q_j^s)} \leqq c \int_{\mathrm{Shad}((\tilde{Q})^-)} f_\alpha^{\#}(y)\, \delta(y)^{\alpha-n} dy$$

$$\leqq c\, |\tilde{Q}|^{\alpha/n}\, \mu(f,x) \leqq c\, \ell(Q_j)^\alpha\, \mu(f,x)$$

since $\tilde{Q} \in \mathcal{Q}$, $\delta(y) \geqq c\, \ell(\tilde{Q})$ when $y \in T(Q_j^s)$, and $|Q_{j_o}| \leqq c\, |Q_j|$ when $j \in J$.

Also since $\operatorname{dist}(Q_j^s, Q_j) \leqq c_o\, \ell(Q_j)$ and $c_o\, \ell(Q_j^s) \geqq \ell(Q_j)$, we have $Q_j^* \subset c(4c_o+1)Q_j^s$. So, using Markov's inequality and Lemma 3.2, we have for any multiindex ν,

(10.21) $$\|D^\nu(P_{Q_j^s} f - P_{\tilde{Q}_o} f)\|_{L_\infty(Q_j^*)} \leqq c\, [\ell(Q_j)]^{\alpha-|\nu|}\, \mu(f,x), \quad j \in J.$$

On the cube R, we have

(10.22) $$\psi := Ef - P_{\tilde{Q}_o} f = \sum_{j\in J} [P_{Q_j^s} f - P_{\tilde{Q}_o} f]\phi_j^*$$

because each ϕ_j^* is supported on Q_j^*. Differentiating any of the terms in the sum (10.22) and using (10.8) and (10.21) together with Leibnitz' rule gives

$$\|D^\nu([P_{Q_j^s} f - P_{\tilde{Q}_o} f]\phi_j^*)\|_{L_\infty(R)} \leqq c\, \ell(Q_j)^{\alpha-|\nu|}\, \mu(f,x).$$

Hence

(10.23) $$\|D^\nu \psi\|_{L_\infty(R)} \leqq c\, \ell(Q_{j_o})^{\alpha-|\nu|}\, \mu(f,x)$$

because $|J| \leqq N$. It follows that ψ is in Lip α on R. Indeed, taking $|\mu| = [\alpha] =: k$, and using (10.23) and that $\ell(R) \leqq \ell(Q_{j_o})$ gives

$$\begin{aligned} |D^\mu\psi(x+h) - D^\mu\psi(x)| &\leqq |h| \sum_{|\nu|=k+1} \|D^\nu\psi\|_{L_\infty(R)} \leqq c\, |h|\, \ell(Q_{j_o})^{\alpha-k-1}\, \mu(f,x) \\ &\leqq c\, h^{\alpha-k}\, \mu(f,x) \end{aligned}$$

whenever $x, x+h \in R$. So the Lip α norm of ψ is at most $c\,\mu(f,x)$. According to Theorem 6.4, there is a polynomial π of degree at most $[\alpha]$ such that

$$\|Ef - (\pi + P_{\tilde{Q}_o} f)\|_{L_\infty(R)} = \|\psi - \pi\|_{L_\infty(R)} \leqq c\, |R|^{\alpha/n}\, \mu(f,x).$$

Integrating over R gives

(10.24) $$\frac{1}{|R|^{\alpha/n+1}} \int_R |Ef - (\pi + P_{\tilde{Q}_o} f)| \leqq c\, \mu(f,x).$$

Therefore the three estimates (10.19), (10.20), and (10.24) together with Lemma 2.1 prove the theorem. □

Let us now briefly describe the case n=1 and Ω an interval which we take to be (0,1). Unions of intervals are handled in the discussion of extensions for domains with minimally smooth boundary in the following section. Let F_c

be the set of intervals I of the form $[-2^{-\nu}, -2^{-\nu-1}]$ or $[1+2^{-\nu-1}, 1+2^{-\nu}]$ for some $\nu \geqq 2$, and associate to such I the interval $I^s := [2^{-\nu-1}, 2^{-\nu}]$ or $I^s := [1-2^{-\nu}, 1-2^{-\nu-1}]$ respectively. Also $I^* := \frac{5}{4}I$. We can enumerate the intervals in $\mathcal{F}_c$ as $\{I_j\}_{j=1}^{\infty}$. This is a covering for $S := (-\frac{1}{4},0) \cup (1,\frac{5}{4})$. Let $\{\phi_j^*\}_{j=1}^{\infty}$ be a partition of unity with the properties (10.8). So, in particular, each ϕ_j^* is supported on I_j^* and $\sum_1^{\infty} \phi_j^* \equiv 1$ on S. The extension operator $E := E_\alpha^{\#}$ is defined by

$$Ef(x) := \begin{cases} f(x), & x \in (0,1) \\ \sum_1^{\infty} P_{I_j^s} f(x)\, \phi_j^*(x), & x \in (-\infty,0) \cup (1,\infty) \end{cases} . \tag{10.25}$$

It follows that Ef vanishes outside of $(-\frac{1}{4},\frac{5}{4})$.

If I is any interval, then $\mathrm{Shad}(I) := I \cap (0,1)$. Defining $\mu(f,x)$ as before with $A_o := 2$, then Theorem 10.4 will hold with the same proof. Without going into detail, let us elaborate on a couple of points of the proof. The geometry is much simpler and in particular one does not need Lemma 10.2. Again, there are three cases to consider in estimating

$$\sup_{I \ni x} \frac{1}{|I|^{\alpha+1}} \inf_{\pi \in \mathbb{P}_{[\alpha]}} \int_I |Ef-\pi| .$$

If $I \subset (0,1)$ the estimate is trivial. If $\mathrm{dist}(I,(0,1)) \leqq 2\,\ell(I)$ but $I \not\subset (0,1)$, then we select an interval $I_o \subset (0,1)$ of the form (o,a) or $(x,1)$ with the properties that $|I_o|$ is the same as the largest interval J which hits I and either $J \subset (0,1) \cap I$ or $J \in \mathcal{F}_c$. Then π can be taken as $P_I f$. The estimate of $\int_{I_o \cap (0,1)} |Ef-P_{I_o} f|$ is trival since Ef=f there. The estimate for $P_{I^s} f - P_{I_o} f$ is done as in (10.14). The third case is when $I \subset [0,1]^c$ and $\mathrm{dist}(I,(0,1)) \geqq 2\,\ell(I)$. We also need only consider $\ell(I) \leqq \frac{1}{8}$ since otherwise $Ef \equiv 0$ on I. It follows that I intersets at most two intervals from $\mathcal{F}_c$ and one can take $\pi := P_{I_o} f$ where I_o is the largest interval from $\mathcal{F}_c$ which hits I. The proof is then the same as in Theorem 10.4.

The following theorem proves that $Ef \in C_p^\alpha(\mathbb{R}^n)$ whenever $f \in C_p^\alpha(\Omega)$, $1 \leqq p \leqq \infty$, and $\alpha > 0$.

Theorem 10.5. Let Ω be an interval in the case $n = 1$ or $\Omega := \{(u,v): u \in \mathbb{R}^{n-1}, v \in \mathbb{R}; v > \phi(u)\}$ in the case $n \geqq 2$ with ϕ in Lip 1. The extension operator $E_\alpha^\#$ defined by (10.25), respectively (10.9), is bounded from $C_p^\alpha(\Omega)$ into $C_p^\infty(\mathbb{R}^n)$, $1 \leqq p \leqq \infty$ with the norm of $E_\alpha^\#$ depending only on α, n, and the Lipschitz constant M. Similiarly, the operators E_k^b are bounded from $\dot{C}_p^k(\Omega)$ into $\dot{C}_p^k(\mathbb{R}^n)$ with norm depending only on k, n, and M.

Proof. Apply an L_p norm to both sides of the inequality in Theorem 10.4 to find

$$\|(Ef)_\alpha^\#\|_{L_p(\mathbb{R}^n)} \leqq c\ [\|\mu(f)\|_{L_p(\mathbb{R}^n)} + \|f_\alpha^\#\|_{L_p(\Omega)}]. \tag{10.26}$$

We now estimate $\|\mu f\|_{L_p(\mathbb{R}^n)}$ by considering the cases $p = 1,\infty$ and then use interpolation.

When $p = \infty$ and $g \in L_\infty$, we have

$$Tg(x) := \sup_{\mathcal{Q}\ni Q\ni x} \left(\frac{1}{|Q|^{\alpha/n+1}} \int_{\mathrm{Shad}(\bar{Q})} |g(y)|[\delta(y)]^\alpha dy\right) \leqq c\ \|g\|_{L_\infty(\Omega)} \tag{10.27}$$

where we used the facts that $\delta(y) \leqq c\ |Q|^{1/n}$, $y \in \mathrm{Shad}(\bar{Q})$, and $|\mathrm{Shad}(\bar{Q})| \leqq c\ |Q|$ when $Q \in \mathcal{Q}$. Recall also that $\mathrm{Shad}(\bar{Q}) \subset \Omega$.

For $p = 1$, we note that $c\ |Q|^{1/n} \geqq \delta(y) + |x-y|$ whenever $x \in Q$, $y \in \mathrm{Shad}(\bar{Q})$ and $Q \in \mathcal{Q}$. Using these facts shows that for $g \in L_1$,

$$Tg(x) \leqq c \int_\Omega |g(y)| \frac{[\delta(y)]^\alpha}{[\delta(y)+|x-y|]^{\alpha+n}}\, dy. \tag{10.28}$$

Applying an L_1 norm to both sides of (10.28) gives

$$\begin{aligned} \|Tg\|_{L_1(\mathbb{R}^n)} &\leqq c \int_\Omega |g(y)|\ \delta(y)^\alpha\ [\textstyle\int_{\mathbb{R}^n}(\delta(y)+|x-y|)^{-\alpha-n}dx]dy \\ &\leqq c \int_\Omega |g(y)|\ \delta(y)^\alpha\ [\delta(y)^{-\alpha}]dy = c\ \|g\|_{L_1(\Omega)}\ . \end{aligned} \tag{10.29}$$

By virtue of (10.27) and (10.29), the sublinear operator T is bounded from $L_\infty(\Omega)$ to $L_\infty(\mathbb{R}^n)$ and $L_1(\Omega)$ to $L_1(\mathbb{R}^n)$. By interpolation T must be bounded from $L_p(\Omega)$ to $L_p(\mathbb{R}^n)$, $1 \leqq p \leqq \infty$, and so since $\mu f \equiv Tf_\alpha^\#$, we have

$$\|\mu f\|_{L_p(\mathbb{R}^n)} \leqq c\ \|f_\alpha^\#\|_{L_p(\Omega)}\ ,\quad 1 \leqq p \leqq \infty.$$

When this is used back in (10.26), we find

$$(10.30)\qquad \|(Ef)^{\#}_{\alpha}\|_{L_p(\mathbb{R}^n)} \le c\, \|f^{\#}_{\alpha}\|_{L_p(\Omega)},\ 1 \le p \le \infty.$$

Finally, we wish to estimate $\|Ef\|_{L_p(\mathbb{R}^n)}$. It follows from the definition of Ef that

$$(10.31)\qquad \|Ef\|^p_{L_p(\mathbb{R}^n)} \le c\, [\|f\|^p_{L_p(\Omega)} + \|\sum_{j=1}^{\infty} P_{Q^s_j} f\, \phi^*_j\|^p_{L_p(\Omega^c)}] .$$

Since each $x \in \Omega^c$ appears in at most N cubes Q^*_j with N depending only on n and M, Hölder's inequality gives

$$|\sum_j P_{Q^s_j} f\, \chi_{Q^*_j}|^p \le N^{p-1} \sum_j |P_{Q^s_j} f\, \chi_{Q^*_j}|^p .$$

Integrating over Ω^c and using the fact that $\lambda Q^s_j \supset Q^*_j$ ($\lambda=4c_o+1$), we get by Lemma 3.2

$$\begin{aligned}\|\sum_j P_{Q^s_j} f\, \chi_{Q^*_j}\|^p_{L_p(\Omega^c)} &\le c \sum_j \|P_{Q^s_j} f\|^p_{L_p(Q^*_j)} \le c \sum_j \|P_{Q^s_j} f\|^p_{L_p(\lambda Q^s_j)}\\ &\le c \sum_j \|P_{Q^s_j} f\|^p_{L_p(Q^s_j)} \le c \sum_j \|f\|^p_{L_p(Q^s_j)}\end{aligned}$$

where the last inequality follows from the fact that $P_{Q^s_j}$ is a bounded operator on $L_p(Q^s_j)$ (see inequality (2.3)). Combining this with equality (10.31) shows that

$$\|Ef\|^p_{L_p(\mathbb{R}^n)} \le c\, [\|f\|^p_{L_p(\Omega)} + \sum_{j=1}^{\infty} \|f\|^p_{L_p(Q^s_j)}].$$

But the Q^s_j coincide for different j at most c_o times, hence

$$\|Ef\|_{L_p(\mathbb{R}^n)} \le c\, \|f\|_{L_p(\Omega)}, \quad 1 \le p \le \infty .$$

Combining this with (10.30) proves the theorem for $E^{\#}_{\alpha}$. Similar reasoning applies for $E^{\flat}_{\alpha}$. □

Remark. The proof of the extension theorem simplifies considerably for the Sobolev spaces $W^k_p(\Omega)$, $1 \le p \le \infty$. First we do not need the cover constructed in Lemma 10.1 and may use instead the standard Whitney coverings F, F_c of both Ω and $\Omega^c \backslash \partial\Omega$, respectively. We let $\{Q_i\}$ be an enumeration of F_c and let $x_i = (u_i, v_i)$ be the center of Q_i. Defining Q^s_i to be that cube in F containing

$x_i^s := x_i + 2\,\delta(x_i)e_n$ where $\delta(x_i) := |\phi(u_i) - v_i|$, we see immediately that properties (10.1)-(10.3) hold. As before, define the extension operator by

$$\text{(10.32)} \qquad Ef(x) := \sum_i \pi_i(x)\, \phi_i^*(x) + f(x)\, \chi_\Omega(x)$$

where π_i is a best $\mathbb{P}_{k-1}$ approximation to f on $L_1(Q_i^s)$ and ϕ_i^* is a partition of unity for the open cover $\{\frac{5}{4}Q_i\}$. For each fixed $x \in \Omega^c \backslash \partial\Omega$ there is a neighborhood U of x which intersects at most $N = 12^n$ of the supports of the ϕ_i's. Let i_o be the index such that $x \in Q_{i_o}$ and define $\bar{Q} := A\, Q_{i_o} + \sqrt{n}MAe_n$ with A large (e.g., $A = 3000M^2$), $\bar{Q} \subset \Omega$, each $Q_i^s \subset \text{Shad}(\bar{Q})$ and $\ell(Q_i^s) \approx \ell(\bar{Q})$ if $U \cap \text{supp}\, \phi_i^* \neq \phi$. It is not too difficult to prove that for

$\mathcal{D}g := \sum_{|\nu|=k} |D^\nu g|$

$$\text{(10.33)} \qquad (Ef) \leq c\; T(\mathcal{D}f) + \mathcal{D}f\, \chi_\Omega$$

where the operator $Tg(x) := \sup_{\mathcal{Q} \ni Q \ni x} \frac{1}{|Q|^{k/n+1}} \int_{\text{Shad}\,\bar{Q}} |g(y)|\delta(y)^k dy$ is bounded on L_1 and L_∞ (see (10.27) and (10.29)). Here $\mathcal{Q} := \{Q: \text{dist}(Q,\partial\Omega) \leq \ell(Q)\}$ and $\text{Shad}\,\bar{Q} = \{(u,v) \in \Omega:\ (u, v_o) \in \bar{Q} \text{ with } v \leq v_o\}$. It follows at once from (10.33) that

$$||Ef||_{W_p^k(\mathbb{R}^n)} \leq c\; ||f||_{W_p^k(\Omega)} .$$

Two main estimates are needed of the proof of (10.33): if $|\nu| = k$, then

$$\text{(10.34)} \qquad |D^\nu(Ef)(x)| \leq \Big| \sum_{0\leq\mu\leq\nu} \binom{\nu}{\mu} \sum_i D^\mu \pi_i(x)\, D^{\nu-\mu}\phi_i^*(x)\Big|$$

$$\leq c \sum_i ||\pi_i - \bar{\pi}||_{L_\infty(Q_i)} \ell(Q_i)^{-k}\, \chi_{Q_i^*}(x)$$

(where $\bar{\pi} := P_{\bar{Q}}^{b} f$, a best $\mathbb{P}_{k-1}$ approximation to f on $L_1(\bar{Q})$) and

$$\text{(10.35)} \qquad ||P_Q^b f - P_{Q^*}^b f||_{L_\infty(Q)} \leq c\; \ell(Q)^{k-n} \int_Q \mathcal{D}f(y)dy$$

if $Q^* \subset Q \subset 4Q^*$ with $Q \in F$.

The first inequality of (10.34) follows by applying Leibnitz' rule while the second follows from the facts that $\bar{\pi} \in \mathbb{P}_{k-1}$ and $D^{\nu-\mu}(\Sigma\phi_i^*) \equiv 0$ if $\mu \neq \nu$, together with Markov's inequality and the estimate $|D^{\nu-\mu}\phi_i^*| \leq c\, \ell(Q_i)^{|\mu|-k}$. Inequality (10.35) follows immediately from Theorem 3.4 with p = 1 and Lemma 3.1. Finally these two estimates are used with the fact that $P^b_{Q_i^s} f - P^b_{\overline{Q}} f$ can be written as a telescoping sum of terms of the type $P^b_Q f - P^b_{Q^*} f$.

§11. Extensions for Domains With Minimally Smooth Boundary

In this section, we piece together the extensions of §10 to give extension operators for more general domains. We first discuss the case $n > 1$ and leave the case $n = 1$ to a remark following Theorem 11.4. The domains of §10 were of the form

$$\Omega = \{(u,v): u \in \mathbb{R}^{n-1},\ v \in \mathbb{R};\ \phi(u) < v\}, \quad |\phi|_{\mathrm{Lip}\ 1} \leq M.$$

We call such a domain: a <u>special Lipschitz domain</u>. Any rotation of such a domain is called a <u>special rotated domain</u>.

Suppose, we are given $\varepsilon_o > 0$, an integer $N_o > 0$, a sequence of open sets $\{U_i\}$, and a sequence of special rotated domains $\{\Omega_i\}$ with the properties:

(11.1) i) if $x \in \partial\Omega$, then $B_{\varepsilon_o}(x) \subset U_i$ for some i

ii) $B_{\varepsilon_o}(x)$ intersects at most N_o sets U_i

iii) for each i, $\Omega \cap U_i = \Omega_i \cap U_i$,

then we say Ω is a domain with <u>minimally smooth boundary</u>. This definition is equivalent[a)] to the usual definition [15, p. 189] which replaces ii) by the requirement: ii)' $\Sigma\, \chi_{U_i} \leq N_o'$. Indeed, if Ω satisfies i), iii), and ii)' for some $(U_i', \varepsilon_o', N_o')$, then the sets $U_i := (U_i')^{2\varepsilon_o}$ with $\varepsilon_o := \varepsilon_o'/4$ and $N_o := N_o'$ satisfy i)-iii) because any sphere $B_{\varepsilon_o}(x_o)$ which intersects U_i satisfies $x_o \in U_i'$.

We now construct a partition of unity as in [15]. For full details of its properties see [15, p. 190-191]. If U is an open set, then $U^{\varepsilon} := \{x \in U: B_{\varepsilon}(x) \subset U\}$. It follows from (11.1) i) that $\{U_i^{\varepsilon_o}\}$ is a cover for $\partial\Omega$. Now, fix $\varepsilon_1 := \varepsilon_o/8$ and define

$$\lambda_i(x) := \chi_{U_i^{2\varepsilon_1}} * \eta_{\varepsilon_1}(x)$$

where η is a C^{∞} function supported on the unit ball and $\eta_{\varepsilon}(x) := \varepsilon^{-n}\,\eta(x/\varepsilon)$ are the dilates of η. Then λ_i is supported on $U_i^{\varepsilon_1}$ and $\lambda_i \equiv 1$ on $U_i^{3\varepsilon_1}$.

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a) For the original proof, see R. Sharpley, "Cone conditions and the modulus of continuity", to appear in the Proceedings of the Second Edmonton Conference on Approximation Theory, CMS Conf. Proc., Vol. 3, AMS, 1983.

Going further, let

$$U_o: = \{x: \ \mathrm{dist}(x,\Omega) < \varepsilon_1\}$$

$$U_+: = \{x: \ \mathrm{dist}(x,\partial\Omega) < 2\varepsilon_1\}$$

$$U_-: = \{x \in \Omega: \ \mathrm{dist}(x,\partial\Omega) > 2\varepsilon_1\}$$

and let λ_o, λ_+ and λ_- be defined as above with $\chi_{\Omega_i^{2\varepsilon_1}}$ replaced by χ_{U_o}, χ_{U_+} and χ_{U_-} respectively. The functions

$$\Lambda_+: = \lambda_o\left(\frac{\lambda_+}{\lambda_+ + \lambda_-}\right) \quad \text{and} \quad \Lambda_-: = \lambda_o\left(\frac{\lambda_-}{\lambda_+ + \lambda_-}\right)$$

satisfy: $\Lambda_+ + \Lambda_- = 1$ on $\bar{\Omega}$.

To define our extension operator, set $\phi_i: = \Lambda_+ \lambda_i / \Sigma\lambda_j^2$. Since $\Sigma\lambda_j^2 \geq 1$ on support of Λ_+, the functions ϕ_i as well as the λ_i, Λ_+ and Λ_- have a uniform bound for their $W_\infty^{[\alpha]+1}$ norms which we denote by L. Finally, define

$$Ef: = \Sigma\ \phi_i\ E_i(\lambda_i f) + \Lambda_- f$$

where for each i, E_i is the extension operator for Ω_i guaranteed by Theorem 10.5. We now proceed to show that Ef is in $C_p^\alpha(\mathbb{R}^n)$ whenever $f \in C_p^\alpha(\Omega)$.

Since rotations are involved in the definition of E, we need to examine the effect of replacing the cubes Q in the definition of $f_\alpha^\#$ by rotated cubes or more general collections of sets. We say that a collection $\mathcal{S}$ of measurable subsets of $\mathbb{R}^n$ is <u>admissable</u> if there is a constant $c' > 0$ such that for each standard cube (sides parallel to the axes), there is an $S \in \mathcal{S}$ with $c'S \subset Q \subset S$ ($c'S$ denotes the set S dilated by c' about its center of gravity) and conversely for each $S \in \mathcal{S}$ there is a standard cube Q with $c'Q \subset S \subset Q$. Examples of admissable collections are balls, finite cones with fixed angle, etc. For our purposes the most important admissable collections are the collection of all cubes and the collection of all cubes which are a fixed rotation of standard cubes.

If $\mathcal{S}$ is an admissable collection and Ω is a domain, let

$$(11.2) \qquad F_\alpha(x): = \sup_{\substack{\Omega\supset S\ni x \\ S\in\mathcal{S}}} \frac{1}{|S|^{1+\alpha/n}} \inf_{\pi\in \mathbb{P}_{[\alpha]}} \int_S |f - \pi|, \qquad x \in \Omega.$$

Lemma 11.1. If Ω is a special rotated domain, $\mathcal{S}$ an admissable collection and $\alpha > 0$, then there are constants $c_1, c_2 > 0$ depending only on α, n, c', and M such that for each $1 \leq p \leq \infty$,

$$(11.3) \qquad c_1 \, |f|_{C_p^\alpha(\Omega)} \leq \|F_\alpha\|_{L_p(\Omega)} \leq c_2 \, |f|_{C_p^\alpha(\Omega)}.$$

Proof. Consider first the case $\Omega = \{(u,v): \phi(u) < v\}$. We will use the results of §10 with the following adjustments on the constants appearing there. First, in the definition of the cone C, we increase the value of M so that whenever $Q \in \mathcal{F}$ then $\frac{1}{c'} Q \subset \Omega$. This is possible since the effect of increasing M is to push the cubes $Q \in F$ further away from $\partial\Omega$. We also increase the constant A_o so that $A_o \geq 2\sqrt{n}/c'$. The results of §10 hold with A_o arbitrarily large.

Now consider the right hand inequality in (11.3). Suppose $x \in S \subset \Omega$ with $S \in \mathcal{S}$ and let R be a standard cube with $c'R \subset S \subset R$. If $A_o|S|^{1/n} \leq \operatorname{dist}(S,\partial\Omega)$, then $A_o c'|R|^{1/n} \leq \operatorname{dist}(S,\partial\Omega) \leq \operatorname{dist}(c'R,\partial\Omega)$. Since $A_o \geq 2\sqrt{n}/c'$, we have $R \subset \Omega$ and

$$(11.4) \qquad \frac{1}{|S|^{1+\alpha/n}} \inf_{\pi \in \mathbb{P}_{[\alpha]}} \int_S |f - \pi| \leq \frac{c}{|R|^{1+\alpha/n}} \inf_{\pi \in \mathbb{P}_{[\alpha]}} \int_R |f - \pi| \leq c \, f_\alpha^\#(x).$$

On the other hand, if $\operatorname{dist}(S,\partial\Omega) \leq A_o \, |S|^{1/n}$ then $\operatorname{dist}(R,\partial\Omega) \leq A_o \ell(R)$ and so by Lemma 10.3

$$(11.5) \qquad \inf_{\pi \in \mathbb{P}_{[\alpha]}} \int_S |f - \pi| \leq \inf_{\pi \in \mathbb{P}_{[\alpha]}} \int_R |E_\Omega f - \pi| \leq c \int_{\operatorname{Shad}(\bar{R})} f_\alpha^\#(y)\delta(y)^\alpha dy$$

where E_Ω is the extension operator for Ω. Hence, if T is the operator defined by (10.27) then

$$(11.6) \qquad \frac{1}{|S|^{1+\alpha/n}} \inf_{\pi \in \mathbb{P}_{[\alpha]}} \int_S |f - \pi| \leq c \, Tf_\alpha^\#(x)$$

when $\operatorname{dist}(S,\partial\Omega) \leq A_o \, |S|^{1/n}$. Combining (11.4) and (11.6) gives

$$(11.7) \qquad F_\alpha(x) \leq c \, [f_\alpha^\#(x) + Tf_\alpha^\#(x)] \qquad x \in \Omega.$$

Since T is bounded on L_p, the right hand inequality in (11.3) follows.

The left hand inequality follows from the estimate

$$(11.8) \qquad f_\alpha^\# \leq c \, [F_\alpha + TF_\alpha],$$

whose proof is much the same as (11.7). Suppose x is in the standard cube $R \subset \Omega$ and $S \in \mathcal{S}$ satisfies $c'S \subset R \subset S$. If $A_o \ell(R) \leqq \mathrm{dist}(R, \partial\Omega)$, then $S \subset \Omega$ and

$$\frac{1}{|R|^{1+\alpha/n}} \inf_{\pi \in \mathbb{P}_{[\alpha]}} \int_R |f - \pi| \leqq c\, F_\alpha(x). \tag{11.9}$$

If $\mathrm{dist}(R, \partial\Omega) \leqq A_o\, \ell(R)$, then we proceed as in Lemma 10.3. Let $Q \in F$ with $Q \cap R \neq \phi$ and let $R_m = Q, R_{m-1}, \ldots, R_1, R_o$ be as in Lemma 10.3. For each j, there is a set $S_j \in \mathcal{S}$ with $c'S_j \subset R_j \subset S_j$ and a polynomial $\pi_j \in \mathbb{P}_{[\alpha]}$ which is a best approximation to f in $L_1(S_j)$. Furthermore $S_j \subset \frac{1}{c'} R_j \subset \Omega$, $j = 0, \ldots, m$. Hence the same telescoping argument which was used in deriving (10.13) together with Lemma 3.2 shows that

$$\|\pi_m - \pi_o\|_{L_\infty(Q)} \leqq c \sum_{j=o}^{m} m_j\, |R_j|^{\alpha/n}$$

with $m_j := \inf_{S_j'} F_\alpha$ and $S_j' := c'S_j$. Using the same technique as in the derivation of (10.16) shows that

$$\int_R |f - \pi_o| \leqq c \int_{\mathrm{Shad}(\bar{R})} F_\alpha(y)\, \delta(y)^{\alpha-n} \psi(y)\,dy \tag{11.10}$$

$$\leqq c \int_{\mathrm{Shad}(\bar{R})} F_\alpha(y)\, \delta(y)^{\alpha}\,dy$$

since $\psi(y) := \sum_{\substack{Q \cap R \neq \phi \\ Q \in F}} |Q|\, \chi_{T(Q)}(y) \leqq c\, \delta(y)^n$ with $T(Q) = \bigcup_{j=o}^{m} R_j$.

From (11.10), it follows that

$$\inf_{\pi \in \mathbb{P}_{[\alpha]}} \frac{1}{|R|^{1+\alpha/n}} \int_R |f - \pi| \leqq c\, TF_\alpha(x).$$

This together with (11.9) establishes (11.8), and therefore verifies (11.3) for domains $\Omega = \{(u,v): \phi(u) < v\}$.

It follows from what we have proved that given any two admissable collections $\mathcal{S}$ and $\mathcal{S}'$ the corresponding maximal functions F_α and F_α' have comparable L_p norms. Thus given any special rotated domain, (11.3) follows by taking an inverse rotation. □

<u>Remark</u>. In the arguments given above and in §10, we could replace $\mathbb{P}_{[\alpha]}$ by $\mathbb{P}_j$, $j \geqq [\alpha]$, and the proofs remain valid for the resulting maximal operators

$$ {}_jf_\alpha(x) := \sup_{\Omega\supset Q\ni x} \left\{\frac{1}{|Q|^{1+\alpha/n}} \inf_{\pi\in\mathbb{P}_j} \int_Q |f-\pi|\right\} $$

$$ {}_jF_\alpha(x) := \sup_{\substack{\Omega\supset S\ni x\\ S\in\mathcal{S}}} \left\{\frac{1}{|S|^{1+\alpha/n}} \inf_{\pi\in\mathbb{P}_j} \int_S |f-\pi|\right\}. $$

In particular, for $j \geqq [\alpha]$, there are constants $c_1, c_2 > 0$ such that

(11.11) $$ c_1 \, ||{}_jf_\alpha||_{L_p(\Omega)} \leqq ||{}_jF_\alpha||_{L_p(\Omega)} \leqq c_2 \, ||{}_jf_\alpha||_{L_p(\Omega)}. $$

The following lemma is in essence a version of Lemma 2.3 for admissable collections.

Lemma 11.2. If Ω is a special rotated domain, $1 \leqq p \leqq \infty$; $\alpha \geqq 0$ and $j \geqq [\alpha]$ then there are c_1, $c_2 > 0$ such that for $f \in L_1(\Omega) + L_\infty(\Omega)$

(11.12) $$ c_1 \, ||f||_{C_p^\alpha} \leqq ||{}_jf_\alpha||_{L_p(\Omega)} + ||f||_{L_p(\Omega)} \leqq ||f||_{C_p^\alpha}. $$

Proof. By Lemma 11.1 and the remark following it, we can assume that Ω is a special Lipschitz domain. The right hand inequality is immediate since $\mathbb{P}_{[\alpha]} \subset \mathbb{P}_j$. For the left hand inequality, take $\mathcal{S}$ to be the collection of all finite cones $\{(u,v) \colon v_o + M|u-u_o| < v \leqq v_o + h\}$ of height h and vertex $x_o = (u_o, v_o)$ and let F_α be as in (11.2). If we use cones $S = S_o \subset S_1 \subset \ldots \subset S_N \subset \Omega$, with $|S_i| = 2^{-n} |S_{i+1}|$, in place of the cubes Q_i in the proof Lemma 2.3 then we find

$$ F_\alpha \leqq c \; {}_jF_\alpha. $$

Using (11.3) and (11.11), we have

$$ ||f||_{C_p^\alpha(\Omega)} \leqq c \, [||F_\alpha||_{L_p(\Omega)} + ||f||_{L_p(\Omega)}] $$
$$ \leqq c \, [||{}_jF_\alpha||_{L_p(\Omega)} + ||f||_{L_p(\Omega)}] \leqq c \, [||{}_jf_\alpha||_{L_p(\Omega)} + ||f||_{L_p(\Omega)}] $$

as desired. □

Remark. The estimate (11.12) holds also for the maximal function ${}_jf_{\alpha,q}$ which is defined in the same manner as ${}_jf_\alpha$ except with L_q "norms", $0 < q \leqq p$, in place of the L_1 norm. For the proof we make modifications similiar to those made in the proof of Lemma 4.4.

Because of the form of the extension operator E, we will have to estimate $(\lambda g)^{\#}_{\alpha}$ when λ is smooth and g is a general function. Suppose λ is supported in an open set U and $\varepsilon > 0$. Let $N_\varepsilon := N_\varepsilon(U)$ denote the ε neighborhood of U.

<u>Lemma 11.3</u>. If Ω is a special rotated domain and $1 \leq p \leq \infty$, then there is a constant c depending only on ε, M, n, p and $||\lambda||_{W^{[\alpha]+1}_{\infty}}$ such that

$$||\lambda f||_{C^{\alpha}_{p}(\Omega)} \leq c \, ||f||_{C^{\alpha}_{p}(N_\varepsilon \cap \Omega)}$$

<u>Proof</u>. Clearly, $||\lambda f||_{L_p(\Omega)} \leq c \, ||f||_{L_p(N_\varepsilon \cap \Omega)}$. Consider first the case $1 < p \leq \infty$. According to Lemma 11.2, it suffices to show

$$||{}_j(\lambda f)_\alpha||_{L_p(\Omega)} \leq c \, ||f||_{C^{\alpha}_{p}(N_\varepsilon \cap \Omega)} \tag{11.13}$$

for $j = 2[\alpha]$. Suppose then that $x \in \Omega$ and Q is a cube satisfying $\Omega \supset Q \ni x$.

If $|Q| \geq \varepsilon^n$, then

$$\inf_{\pi \in \mathbb{P}_j} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |\lambda f - \pi| \leq c \, M(f\chi_{\Omega \cap U})(x) \leq c \, M(f\chi_{N_\varepsilon \cap \Omega})(x). \tag{11.14}$$

If $|Q| \leq \varepsilon^n$, then we may assume $Q \cap U \neq \phi$ since otherwise $\lambda f \chi_Q \equiv 0$. Let π_o and π_λ denote best $L_1(Q)$ approximations from $\mathbb{P}_{[\alpha]}$ to f and λ respectively. Writing $\lambda f - \pi_\lambda \pi_o = (f - \pi_o)\lambda + \pi_o(\lambda - \pi_\lambda)$, we have

$$\begin{aligned} \inf_{\pi \in \mathbb{P}_j} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |\lambda f - \pi| &\leq c \, f^{\#}_{\alpha}(x) + ||\pi_o||_{L_\infty(Q)} \lambda^{\#}_{\alpha}(x) \\ &\leq c \, [f^{\#}_{\alpha}(x) + M(f\chi_{N_\varepsilon \cap \Omega})(x)] \end{aligned} \tag{11.15}$$

where we used the facts that $\lambda \in W^{[\alpha]+1}_{\infty}$, $||\pi_o||_{L_\infty(Q)} \leq \frac{c}{|Q|} \int_Q |f|$ and $Q \subset N_\varepsilon$. In this inequality $f^{\#}_{\alpha}$ is taken relative to the domain $N_\varepsilon \cap \Omega$. Inequality (11.13) follows easily from (11.14) and (11.15) because M is bounded on L_p.

When $p = 1$, we choose $(1+\frac{\alpha}{n})^{-1} < q < 1$ and use $f^{\#}_{\alpha,q}$ in place of $f^{\#}_{\alpha}$ (see Theorem 4.3) and M_q in place of M to derive an analogous inequality to (11.13) with ${}_jf_{\alpha,q}$ in place of ${}_jf_\alpha$. □

We can now prove the main result of this section.

Theorem 11.4. Suppose Ω is a domain with minimally smooth boundary. For each $\alpha > 0$, and $1 \leqq p \leqq \infty$,

$$\|Ef\|_{C_p^\alpha(\mathbb{R}^n)} \leqq c\, \|f\|_{C_p^\alpha(\Omega)} \tag{11.16}$$

with c depending only on α, n, and Ω.

Proof. Consider first the case $1 < p < \infty$. Let $g_i := \phi_i E_i(\lambda_i f)$ and $g_o := \Lambda_- f$. Then,

$$\|Ef\|_{C_p^\alpha(\mathbb{R}^n)} \leqq \|\Sigma g_i\|_{C_p^\alpha(\mathbb{R}^n)} + \|g_o\|_{C_p^\alpha(\mathbb{R}^n)}. \tag{11.17}$$

First, we estimate the term involving g_o. Since Λ_- is supported on Ω^{ε_1}, for any cube Q with $x \in Q$ and $|Q| \geqq (\varepsilon_1/\sqrt{n})^n$ we have

$$\inf_{\pi \in \mathbb{P}_{[\alpha]}} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |g_o - \pi| \leqq c\, M(f\chi_\Omega)(x).$$

On the other hand if $x \in Q$ and $|Q| < (\varepsilon_1/\sqrt{n})^n$, then we can estimate as in (11.15) and obtain

$$\begin{aligned} \|g_o\|_{C_p^\alpha(\mathbb{R}^n)} &\leqq c\, [\|f\|_{C_p^\alpha(\Omega)} + \|M(f\chi_\Omega)\|_{L_p(\mathbb{R}^n)}] \\ &\leqq c\, \|f\|_{C_p^\alpha(\Omega)} \end{aligned} \tag{11.18}$$

because $p > 1$.

To estimate the term involving Σg_i in (11.17), we again consider the case $|Q| \geqq (\varepsilon_1/\sqrt{n})^n$ and find

$$\begin{aligned} \inf_{\pi \in \mathbb{P}_{[\alpha]}} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |(\Sigma g_i) - \pi| &\leqq c\, M(Ef - g_o)(x) \\ &\leqq c\, [M(Ef)(x) + M(f\chi_\Omega)(x)]. \end{aligned} \tag{11.19}$$

If $|Q| < (\varepsilon_1/\sqrt{n})^n$ and $x \in Q$, then Q intersects at most N_o of the $U_i^{\varepsilon_1}$. We denote by $I := I(Q)$ the set of such indices i. For $i \in I(Q)$ let π_i denote a best $L_1(Q)$ approximation to g_i from $\mathbb{P}_{[\alpha]}$ and set $\pi := \sum_{i \in I} \pi_i$. Then, $Q \subset U_i$ and so

$$(11.20)\quad \frac{1}{|Q|^{1+\alpha/n}} \int_Q |(\Sigma g_i) - \pi| \le \frac{1}{|Q|^{1+\alpha/n}} \sum_{i\in I} \int_Q |g_i - \pi_i|$$
$$\le \sum_{i\in I} (g_i)_\alpha^{\#}(x)\, \chi_{U_i}(x).$$

This, together with (11.19) gives

$$(11.21)\quad (\Sigma g_i)_\alpha^{\#}(x) \le c\, [M(Ef)(x) + M(f\chi_\Omega)(x) + \sum_{i\in I} (g_i)_\alpha^{\#}(x)\chi_{U_i}(x)].$$

Concentrating on the last term, we notice that

$$(11.22)\quad |\sum_{i\in I}(g_i)_\alpha^{\#}(x)\chi_{U_i}(x)|^p \le N_o^{p-1}\, \Sigma(g_i)_\alpha^{\#}(x)^p$$

because $\Sigma\chi_{U_i}(x) \le N_o$. Using this in (11.21) gives

$$(11.23)\quad ||(\Sigma g_i)_\alpha^{\#}||_{L_p}^p \le c\, [||M(Ef)||_{L_p}^p + ||M(f\chi_\Omega)||_{L_p}^p + \Sigma\, ||(g_i)_\alpha^{\#}||_{L_p}^p].$$

But M is bounded on L_p for $p > 1$ and E: $L_p(\Omega) \to L_p(\mathbb{R}^n)$ and so

$$(11.24)\quad ||M(Ef)||_{L_p}^p + ||M(f\chi_\Omega)||_{L_p}^p \le c\, ||f||_{L_p(\Omega)}^p$$

For each i, Lemma 11.3 (with $\Omega = \mathbb{R}^n$) and Theorem 10.5 give

$$||(g_i)_\alpha^{\#}||_{L_p} \le c\, ||E_i(\lambda_i f)||_{C_p^\alpha} \le c\, ||\lambda_i f||_{C_p^\alpha(\Omega_i)}.$$

This time applying Lemma 11.3 to $\lambda_i f$ with $U = U_i^{\varepsilon_1}$ and using the fact that $N_{\varepsilon_1}(U) \subset U_i$, we have

$$||(g_i)_\alpha^{\#}||_{L_p}^p \le c\, ||f||_{C_p^\alpha(U_i\cap\Omega_i)}^p \le c \int_{U_i\cap\Omega} (f_{\alpha,\Omega}^{\#} + |f|)^p$$

because $U_i \cap \Omega_i = U_i \cap \Omega$. Since each x appears in at most N_o U_i's, substituting this and (11.24) back into (11.23), gives

$$||(\Sigma g_i)_\alpha^{\#}||_{L_p} \le c\, [||f||_{L_p(\Omega)} + ||f_\alpha^{\#}||_{L_p(\Omega)}] \le c\, ||f||_{C_p^\alpha(\Omega)}.$$

Also as noted above

$$||\Sigma g_i||_{L_p} \le ||Ef||_{L_p} + ||f\chi_\Omega||_{L_p} \le c\, ||f||_{L_p(\Omega)}.$$

This completes the proof for $1 < p < \infty$.

For $p = \infty$, we use $\sum_{i\in I} (g_i)_\alpha^{\#}(x)\, \chi_{U_i}(x) \le N_o\, \Sigma(g_i)_\alpha^{\#}(x)$ in place of (11.22). Then, the same proof with L_∞ norms in place of L_p norms gives the desired result. For $p = 1$, we choose $(1+\frac{\alpha}{n})^{-1} < q < 1$ and use $f^{\#}_{\alpha,q}$ in place of $f^{\#}_{\alpha}$ and $M_q f$ in place of Mf with the same proof and the fact that $||h^{\#}_{\alpha,q}||_{L_1(\mathcal{O})} \leq c\, ||h^{\#}_{\alpha}||_{L_1(\mathcal{O})}$ for any $\mathcal{O}$ with c independent of $\mathcal{O}$ (see Theorem (4.3)). □

Remarks.

i) The extension theorem holds for the spaces $\dot{C}^{\alpha}_{p}$, $1 \leq p \leq \infty$. When α is not an integer, this follows from the fact that $\dot{C}^{\alpha}_{p} = C^{\alpha}_{p}$. When α is an integer, it follows from the argument on page 192 of [15] and the Remark on Sobolev spaces at the end of §10. The space $\dot{C}^{k}_{1}$ must be handled separately using the techniques of this section.

ii) The extension operator E can easily be modified so that for a fixed k, $E: C^{\alpha}_{p}(\Omega) \to C^{\alpha}_{p}(\mathbb{R}^n)$, for all $\alpha < k$. Notice however that it is not a total extension operator in the sense of [15].

iii) The extension theorem holds for domains $\Omega \subset \mathbb{R}$ such that $\Omega = \bigcup_i I_i$ with the I_i intervals satisfying: $\mathrm{dist}(I_i, I_j) \geq \varepsilon_o$, $i \neq j$ and $\ell(I_i) \geq \varepsilon_o$. Here one simply works with a standard partition of unity rather than the more complicated partition used for $n > 1$.

We can now generalize the results of the previous sections which held for special domains to domains with minimally smooth boundary. Maximal functions based on admissable collections rather than cubes can be shown to give equivalent norms for $C^{\alpha}_{p}(\Omega)$.

The interpolation theorems of §8 hold for domains Ω with minimally smooth boundary. For example, it follows immediately from Theorem 11.4 together with Corollary 8.3 that $C^{\alpha}_{p}(\Omega)$ is an interpolation space for $C^{\alpha}_{p_o}(\Omega)$ and $C^{\alpha}_{p_1}(\Omega)$, $p_o < p < p_1$. Going further, one can prove in a similar way to Theorem 11.4 and the generalization of Lemma 11.2 that the interpolation results (8.10) and (8.19) hold.

We also have the following embeddings.

Corollary 11.5. If Ω is a domain with minimally smooth boundary, $0 < p \leqq q \leqq \infty$, and $0 \leqq \beta \leqq \alpha + n(\frac{1}{q}-\frac{1}{p})$, then we have the continuous embeddings

$$C_p^\alpha(\Omega) \to C_q^\beta(\Omega).$$

Proof. Let E be an extension operator for α and Ω. For any $\Omega \supset Q \ni x$ and $\pi \in \mathbb{P}_{[\alpha]}$,

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |f - \pi| \leqq \frac{1}{|Q|^{1+\beta/n}} \int_Q |Ef - \pi|$$

thus,

$$\|f\|_{C_q^\beta(\Omega)} \leqq \|Ef\|_{C_q^\beta(\mathbb{R}^n)}. \tag{11.25}$$

From Theorems 9.6 and 11.4,

$$\|Ef\|_{C_q^\beta(\mathbb{R}^n)} \leqq c\,\|Ef\|_{C_p^\alpha(\mathbb{R}^n)} \leqq c\,\|f\|_{C_p^\alpha(\Omega)}$$

which together with (11.25) proves the Corollary. □

We can also generalize the results of Theorem 7.1. Here, we use the fact that

$$(L_p(\Omega),\ W_p^k(\Omega))_{\theta/k,q} = B_p^{\theta,q}(\Omega). \tag{11.26}$$

This was proved for domains Ω which satisfy a uniform cone condition in [11].[b)]

Corollary 11.6. If Ω is a domain with minimally smooth boundary, then for $1 < p < \infty$, we have the continuous embeddings

$$B_p^{\alpha,p}(\Omega) \to C_p^\alpha(\Omega) \to B_p^{\alpha,\infty}(\Omega).$$

Proof. Let $k > \alpha$. For the right hand embedding, let E be the extension operator for k and Ω, then using Theorem 7.1 and the Remark ii), we have

$$\|f\|_{B_p^{\alpha,\infty}(\Omega)} \leqq \|Ef\|_{B_p^{\alpha,\infty}(\mathbb{R}^n)} \leqq c\,\|Ef\|_{C_p^\alpha(\mathbb{R}^n)} \leqq c\,\|f\|_{C_p^\alpha(\Omega)}.$$

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b) Ibid. This condition is actually equivalent to requiring Ω to have a minimally smooth boundary.

For left hand embedding, we use the fact that E: $B_p^{\alpha,p}(\Omega) \to B_p^{\alpha,p}(\mathbb{R}^n)$ because of (11.26). Using Theorem 7.1, we have

$$||f||_{C_p^\alpha(\Omega)} \leqq ||Ef||_{C_p^\alpha(\mathbb{R}^n)} \leqq c\ ||Ef||_{B_p^{\alpha,p}(\mathbb{R}^n)} \leqq c\ ||f||_{B_p^{\alpha,p}(\Omega)} . \quad \square$$

§12. The case $0 < p < 1$

We want now to define spaces C_p^α and $\mathcal{C}_p^\alpha$ when $0 < p < 1$. We have purposefully postponed the discussion of this case in order to avoid certain technicalities which would only have obscured the development. As we shall see, many of the results of the previous sections hold for this range of p as well.

If $0 < p < 1$ and $\alpha > 0$, let $C_p^\alpha: = C_p^\alpha(\Omega): = \{f \in L_p(\Omega): f_{\alpha,p}^\# \in L_p(\Omega)\}$ and $\mathcal{C}_p^\alpha: = \mathcal{C}_p^\alpha(\Omega): = \{f \in L_p(\Omega): f_{\alpha,p}^\flat \in L_p(\Omega)\}$ and define

$$|f|_{C_p^\alpha} : = ||f_{\alpha,p}^\#||_{L_p} \qquad |f|_{\mathcal{C}_p^\alpha} : = ||f_{\alpha,p}^\flat||_{L_p}$$

$$||f||_{C_p^\alpha} : = ||f||_{L_p} + |f|_{C_p^\alpha} \qquad ||f||_{\mathcal{C}_p^\alpha} : = ||f||_{L_p} + |f|_{\mathcal{C}_p^\alpha}$$

It follows that $d(f,g)_{C_p^\alpha} : = ||f-g||_{C_p^\alpha}^p$ is a metric on C_p^α and $d(f,g)_{\mathcal{C}_p^\alpha} : = ||f-g||_{\mathcal{C}_p^\alpha}^p$ is a metric on $\mathcal{C}_p^\alpha$.

These spaces are F-spaces with respect to their topologies. For example, the proof of the completeness of C_p^α is the same as in the case $p \geq 1$ described in Lemma 6.1. In this case, the inequality

$$h_{\alpha,p}^\#(x) \leq \varliminf_{m\to\infty} (h_m)_{\alpha,p}^\#(x)$$

whenever $h_m \to h$ in L_p follows from the fact that $P_Q h_m \to P_Q h$, which in turn is a consequence of the continuity of P_Q on L_p.

The definitions of C_p^α and $\mathcal{C}_p^\alpha$ for $0 < p < 1$ are consistent with the case $p \geq 1$. Indeed, as we have observed earlier, when $1 \leq p \leq \infty$, Theorem 4.3 shows that

$$f_\alpha^\# \leq f_{\alpha,p}^\# \leq M_\sigma(f_\alpha^\#) \qquad \sigma: = (\frac{1}{p} + \frac{\alpha}{n})^{-1} .$$

Since M_σ is bounded on L_p,

$$||f_\alpha^\#||_{L_p} \leq ||f_{\alpha,p}^\#||_{L_p} \leq c\ ||f_\alpha^\#||_{L_p}$$

and therefore C_p^α could have equivalently been defined as the set of $f \in L_p$ such that $f^{\#}_{\alpha,p} \in L_p$; in addition, $||f^{\#}_{\alpha,p}||_{L_p}$ is equivalent to $|f|_{C_p^\alpha}$.

Suppose now that $\Omega = \mathbb{R}^n$. We want to give embeddings between C_p^α, $0 < p < 1$, and other smoothness spaces. Recall that when $f \in L_p$, $\lim_{Q\downarrow\{x\}} P_Q f(x) = f(x)$, a.e. (Lemma 4.1), and (see (4.10))

$$|P_Q f(x) - f(x)| \leq c\ |Q|^{\alpha/n}\ f^{\#}_{\alpha,p}(x), \quad \text{a.e.} \quad x \in Q. \tag{12.1}$$

Here $P_Q f$ a the best $L_p(Q)$ approximation to f from $\mathbb{P}_{[\alpha]}$. It follows from (12.1) that if $r > [\alpha]$,

$$\Delta_h^r\ (f,x) \leq c\ h^\alpha \sum_{j=1}^{r} f^{\#}_{\alpha,p}(x + jh).$$

Raising both sides to the p-th power and integrating gives the continuous embeddings

$$\mathcal{C}_p^\alpha \to C_p^\alpha \to B_p^{\alpha,\infty} \tag{12.2}$$

with $B_p^{\alpha,q}$ the Besov spaces as defined in §3.

The embeddings

$$B_p^{\alpha,p} \to C_p^\alpha \quad , \quad \alpha > 0, \tag{12.3}$$

also hold for $0 < p < 1$ but their proof requires a litte more care. Let us first consider the case $0 < \alpha < 1$, where there is a simple proof that encompasses the main ideas of the general case. Using Corollary 5.4 and Remark (2.14) i), we have for $Q_\rho := [-\rho,\rho]^n$,

$$\begin{aligned} f^{b}_{\alpha,p}(x) &\leq c \sup_{\rho>0} \frac{1}{\rho^\alpha} \Big(\frac{1}{\rho^n} \int_{Q_\rho} |f(x+s) - f(x)|^p ds\Big)^{1/p} \\ &\leq c \int_0^\infty \frac{1}{\rho^\alpha} \Big(\frac{1}{\rho^n} \int_{Q_\rho} |f(x+s) - f(x)|^p ds\Big)^{1/p} \frac{d\rho}{\rho} \\ &\leq c \sum_{j=-\infty}^{\infty} 2^{-j\alpha}(2^{-jn} \int_{Q_{2^j}} |f(x+s) - f(x)|^p ds)^{1/p} \end{aligned} \tag{12.4}$$

because $\int_{Q_\rho}$ is increasing with ρ. Recall that for $0 < p < 1$, $(\Sigma\lambda_j)^p \leq \Sigma(\lambda_j)^p$.

Hence (12.4) gives

(12.5) $$\int_{\mathbb{R}^n} |f^{b}_{\alpha,p}|^p \leq c \sum_{j=-\infty}^{\infty} 2^{-j\alpha p} \left(2^{-jn} \int_{Q_{2^j}} \int_{\mathbb{R}^n} |f(x+s) - f(x)|^p dx ds\right)$$
$$\leq c \int_0^\infty [\rho^{-\alpha}\, \omega(f,\rho)_p]^p \frac{d\rho}{\rho}$$

and (12.3) readily follows since $f^{b}_{\alpha,p} = f^{\#}_{\alpha,p}$ for $0 < \alpha < 1$.

The case $\alpha \geq 1$ is more involved. Let Q be a cube in $\mathbb{R}^n$ with the same notation as above, we define for $\tau > 0$,

(12.6) $$w_r(f,\tau)_{L_p(Q)} := \left(\tau^{-n} \int_Q \int_{Q_\tau} |\Delta_s^r(f,x)|^p ds dx\right)^{1/p} .$$

For our next lemma, we fix $Q = Q_0$ as the unit cube in $\mathbb{R}^n$ and define S_α as the set of functions in $L_p(\mathbb{R}^n)$ such that

$$||f||_{L_p(a_r Q)} + \sup_{\tau\leq 1} \tau^{-\alpha}\, w_r(f,\tau)_{L_p(a_r Q)} \leq 1$$

where $r := [\alpha] + 1$ and $a_j := 1 + \ldots + j$ for each postive integer j.

<u>Lemma 12.1</u>. For each $\alpha > 0$, S_α is a compact subset of $L_p(Q)$.

<u>Proof</u>. Consider first the case $0 < \alpha < 1$. If m is any postive integer, take $\tau = 1/m$ and subdivide Q into m^n cubes (Q_j) which have pairwise disjoint interiors and each Q_j has side length τ. If $f \in S_\alpha$,

$$\sum_j \int_{Q_j} \int_{Q_\tau} |f(x+s) - f(x)|^p ds dx \leq \tau^{p\alpha+n}.$$

It follows that for each j there is a constant c_j (for example $c_j = f(x_j)$ with appropriately chosen $x_j \in Q_j$) such that the function $\phi_\tau := \sum c_j \chi_{Q_j}$ satisfies

$$\int_Q |f - \phi_\tau|^p \leq \tau^{p\alpha+n} .$$

It is clear that the c_j can be chosen as best constants of approximation to f in $L_p(Q_j)$ and therefore we also have

$$\int_Q |\phi_\tau|^p \leq \int_Q |f - \phi_\tau|^p + \int_Q |f|^p \leq 2 \int_Q |f|^p.$$

Since the span $\{\chi_{Q_j}\}$ is a finite dimensional space and τ can be made arbitrarily small, the set S_α is compact.

The case $\alpha \geq 1$ can be reduced to the case just proved. We start with the identity [19, p. 105]

$$\Delta_s^k(f,x) = 2^{-k} [\Delta_{2s}^k(f,x) - \sum_{j=0}^{k-1} \sum_{i=j+1}^{k} \binom{k}{i}\Delta_s^{k+1}(f,x+js)] .$$

With the abbreviated notation $w_j(\tau) := w_j(f,\tau)_{L_p(a_jQ)}$, we have for $\tau < 1$

$$w_k(\tau)^p \leqq 2^{-kp} w_k(2\tau)^p + c\, w_{k+1}(\tau)^p. \tag{12.7}$$

Since $\tau^n w_k(\tau)^p$ is increasing with τ and $w_k(1) \leqq c\, ||f||_{L_p(a_{k+1}Q)}$, a repeated application of (12.7) gives

$$w_k(\tau)^p \leqq c\, \tau^{kp} [\int_\tau^1 t^{-kp} w_{k+1}(t)^p \frac{dt}{t} + ||f||^p_{L_p(a_{k+1}Q)}] \tag{12.8}$$

with c depending only on k and p.

Now suppose $f \in S_\alpha$ with $r-1 \leqq \alpha < r$. Let $r-2 \leqq \beta < r-1$ and use (12.8) with $k = r-1$ to find

$$w_k(\tau)^p \leqq c\, [\tau^{\beta p} + \tau^{kp}\, ||f||^p_{L_p(a_{k+1}Q)}], \qquad \tau \leqq 1.$$

Hence for an appropriate constant λ, we have $\lambda S_\alpha \subset S_\beta$. Repeated application of this result shows that $\lambda S_\alpha \subset S_{1/2}$ for an appropriate λ. Since $S_{1/2}$ is compact and S_α is closed, we have S_α compact. □

Lemma 12.2. Let $\alpha > 0$; $p > 0$, and $r = [\alpha] + 1$. If $f \in L_p(\mathbb{R}^n)$, then for each cube Q of side length ρ there is a polynomial $\pi_Q \in \mathbb{P}_{r-1}$ such that

$$||f-\pi_Q||_{L_p(Q)} \leqq c\, \rho^\alpha \sup_{\tau\leqq\rho} \tau^{-\alpha} w_r(f,\tau)_{L_p(a_rQ)} \tag{12.9}$$

with $a_r := \frac{1}{2}r(r+1)$.

Proof. The proof is similar to the proofs of Theorem 3.4 and 3.5. It is enough to prove (12.9) for the unit cube since the case of general cubes then follows by scaling. Now, suppose (12.9) does not hold for $Q = Q_0$. It follows that there is a sequence of functions (f_m) such that

(12.10) i) $\operatorname{dist}(f_m, \mathbb{P}_{r-1})_{L_p(Q)} = ||f_m||^p_{L_p(Q)} = 1$

ii) $\sup_{\tau\leqq 1} \tau^{-\alpha} w_r(f_m,\tau)_{L_p(a_rQ)} \to 0 \qquad m \to \infty.$

By Lemma 12.1, (f_m) is precompact in $L_p(Q)$. Hence, we can also assume that $f_m \to f$ in $L_p(Q)$ for some f. For each $0 < \tau < 1$, we have from (12.10) ii),

$$\int_{a_rQ}\int_{Q_\tau} |\Delta_s^r(f,x)|^p ds dx \leq \lim_{m\to\infty} \int_{a_rQ}\int_{Q_\tau} |\Delta_s^r(f_m,x)|^p ds dx = 0. \tag{12.11}$$

Hence it follows that $f = P$ a.e. for some $P \in \mathbb{P}_{r-1}$. On the other hand, (12.10) i) shows that $\operatorname{dist}(f, \mathbb{P}_{r-1}) = 1$ which is the desired contradition. □

Actually when $p < 1$ in the above proof, it may not be so clear that (12.11) implies that $f = P$ a.e. with $P \in \mathbb{P}_{r-1}$. However, this can be proved by induction on r. The case $r = 1$ is obvious. If $r > 1$ and (12.11) holds, then for all sufficiently small s we have $\Delta_s^r(f,x) = 0$ a.e. x. Now we can write (see [11][c)]) a general difference $\Delta_{t_1} \dots \Delta_{t_r}$ in terms of pure differences $\{\Delta_{t_i}^r\}$; hence for all sufficiently small $(t_1,\dots t_r)$, $\Delta_{t_1} \dots \Delta_{t_r}(f,x) = 0$ a.e. in x. Our induction hypothesis then gives that for small t, $\Delta_t(f,x)$ is a.e. a polynomial in $\mathbb{P}_{r-2}$, and therefore it is not difficult to see that

$$f(x+t) = f(x) + \sum_{|\nu|\leq r-2} a_\nu(t)x^\nu \quad \text{a.e. } x \tag{12.12}$$

with a_ν continuous. Applying now an arbitrary r-th difference Δ_s^r to (12.12) as a function of t gives that each $a_\nu(t)$ is a polynomial of degree at most $r - 1$. Taking finally $x = x_0$ such that both (4.7) and (12.12) hold shows that $f = P$ a.e. with $P \in \mathbb{P}_{r-1}$.

The following are embedding theorems for Besov spaces and C_p^α when $p < 1$.

<u>Theorem 12.3</u>. If $\alpha, p > 0$, we have the continuous embeddings

$$B_p^{\alpha,p}(\mathbb{R}^n) \to C_p^\alpha(\mathbb{R}^n) \to B_p^{\alpha,\infty}(\mathbb{R}^n).$$

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c) See also Theorem 1 in B. Baishanski, "The asymptotic behavior of the n-th order difference", Enseignement Mathematique <u>15</u> (1969), 29-41.

Proof. We have shown the right hand embedding in (12.2). The left hand embedding has been shown for $0 < \alpha < 1$ and all $p > 0$ and also for all $\alpha > 0$ provided $p \geqq 1$. Consider now the case $\alpha > 0$; $0 < p < 1$. Choose any $r-2 \leqq \beta < r-1$ (recall $r = [\alpha] + 1$) and let

$$\phi(\rho,x) := \sup_{\tau\leqq\rho} \tau^{-\beta} w_r(f,\tau)_{L_p(x+a_rQ_\rho)}.$$

From Lemma 12.2 and Remark (2.14) i), we have

$$f^{\#}_{\alpha,p}(x) \leqq c \sup_{\rho>0} \rho^{(\beta-\alpha-n/p)} \phi(\rho,x) \leqq c \int_0^\infty \rho^{(\beta-\alpha-n/p)} \phi(\rho,x)\frac{d\rho}{\rho} .$$

Integrating this inequality gives (cf. (12.4-5))

$$\int_{\mathbb{R}^n} |f^{\#}_{\alpha,p}|^p \leqq c \int_0^\infty \rho^{(\beta-\alpha)p} \left(\rho^{-n} \int_{\mathbb{R}^n} \phi(\rho,x)^p dx\right)\frac{d\rho}{\rho} . \tag{12.13}$$

Now,

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(\rho,x)^p \frac{dx}{\rho} &= \int_{\mathbb{R}^n} \left(\sup_{\tau\leqq\rho} \tau^{-\beta p-n} \int_{a_rQ_\rho} \int_{Q_\tau} |\Delta^r_s(f,y-x)|^p ds dy\right) dx \\ &\leqq c \int_{\mathbb{R}^n} \int_0^\rho \left(\tau^{-\beta p-n} \int_{a_rQ_\rho} \int_{Q_\tau} |\Delta^r_s(f,y-x)|^p ds dy\right) \frac{d\tau}{\tau} dx \\ &\leqq c\, \rho^n \int_0^\rho \tau^{-\beta p}\, \omega_r(f,\tau)_p^p \frac{d\tau}{\tau}\end{aligned}$$

where we used the fact that $\omega_r(f,\sqrt{n}\tau)_p \leqq c\, \omega_r(f,\tau)_p$. Returning to (12.13), we have from Hardy's inequality

$$\begin{aligned}\int_{\mathbb{R}^n} |f^{\#}_{\alpha,p}|^p &\leqq c \int_0^\infty \rho^{(\beta-\alpha)p} \int_0^\rho \tau^{-\beta p}\, \omega_r(f,\tau)_p^p \frac{d\tau}{\tau} \frac{d\rho}{\rho} \\ &\leqq c \int_0^\infty \rho^{-\alpha p}\, \omega_r(f,\rho)_p^p \frac{d\rho}{\rho}\end{aligned}$$

as desired. □

The spaces C^α_p, $0 < p \leqq \infty$, form an interpolation scale as is contained in the following generalization of Theorem 8.2.

Theorem 12.4. If $\alpha > 0$ and $0 < p < \infty$,

$$K(f,t,C^\alpha_p,C^\alpha_\infty) \approx \left(\int_0^{t^p} [f^* + f^{\#*}_{\alpha,p}]^p\right)^{1/p} , \quad t > 0$$

$$K(f,t,\dot{C}^\alpha_p,\dot{C}^\alpha_\infty) \approx \left(\int_0^{t^p} [f^* + f^{\flat*}_{\alpha,p}]^p\right)^{1/p} , \quad t > 0 .$$

In addition, if $1/r = (1-\theta)/p + \theta/q$ with $0 < \theta < 1$, then

$$(C_p^\alpha, C_q^\alpha)_{\theta,r} = C_r^\alpha \ ; \quad (\dot{C}_p^\alpha, \dot{C}_q^\alpha)_{\theta,r} = \dot{C}_r^\alpha \ .$$

Proof. The proof of this theorem is much the same as the proof of the case $p = 1$ given in §8. We indicate only the basic changes that have to be made. The projections P_Q are replaced by $\mathcal{P}_Q$ so that $\mathcal{P}_Q f$ is a best $L_p(Q)$ approximant to f of degree $[\alpha]$ in the case of $f_{\alpha,p}^{\#}$ and of degree (α) in the case of $f_{\alpha,p}^{b}$. The extension g of Lemma 8.1 is now defined as

$$g(x): = \begin{cases} f(x), & x \in F \\ \sum_j \mathcal{P}_{Q_j} f(x)\phi_j^*(x), & x \in F^c \end{cases} .$$

The role of the Hardy-Littlewood maximal operator M is replaced by M_p and of course f_α is replaced by $f_{\alpha,p}$ which is either $f_{\alpha,p}^{\#}$ or $f_{\alpha,p}^{b}$ as appropriate. Lemma 8.1 then reads: If $M_p f \leq m_0$ and $f_{\alpha,p} \leq m_1$ on F then i) $g = f$ on F; ii) $g \leq c\, m_0$ on $\mathbb{R}^n$; and iii) $g_{\alpha,p} \leq c\, m_1$ on $\mathbb{R}^n$.

The proofs of Lemma 8.1 and Theorem 8.2 require estimates for $\mathcal{P}_Q f - \mathcal{P}_{Q^*} f$ when $Q^* \subset Q$. We have from (5.5)

$$\|D^\nu(\mathcal{P}_Q f - \mathcal{P}_{Q^*} f)\|_{L_\infty(Q^*)} \leq c\ |Q|^{(\alpha-|\nu|)/n} \inf_{u \in Q^*} f_{\alpha,p}(u). \tag{12.14}$$

This is used in (8.5) with $\nu = 0$ and in the derivation of (8.8) and (8.9).

In the proof of Theorem 8.2, the set E is now defined by

$$E: = \{f_{\alpha,p}^{\#} > f_{\alpha,p}^{\#*}\ (t^p)\} \cup \{M_p f > (M_p f)^*(t^p)\}$$

so that $|E| \leq c\ t^p$. Then, (8.12) becomes

$$t\ \|g\|_{C_\infty^\alpha} \leq c\ (\textstyle\int_0^{t^p} [f^* + f_{\alpha,p}^{\#*}]^p)^{1/p}.$$

On $\tilde{E}$, the estimate (8.15) becomes

$$\int_{\tilde{E}} [h_{\alpha,p}^{\#}]^p \leq c \int_0^{t^p}\ [f_{\alpha,p}^{\#*}]^p$$

and on $\tilde{F}$ (8.17) becomes,

$$h_{\alpha,p}^{\#}(x) \leq c\ f_{\alpha,p}^{\#*}\ (t^p) \left(\sum_j \frac{|Q_j|^{1+\alpha p/n}}{\operatorname{dist}(x,Q_j)^{n+\alpha p}} \right)^{1/p} , \quad x \in \tilde{F}$$

and so

$$\int_{\mathbb{R}^n} [h^{\#}_{\alpha,p}]^p \leq c \sum_j |Q_j| [f^{\#*}_{\alpha,p}(t^p)]^p \leq c\ t^p\ [f^{\#*}_{\alpha,p}(t^p)]^p \leq c \int_0^{t^p} [f^{\#*}_{\alpha,p}]^p\ .$$

This combines with the above inequality for g to give

$$K(f,t,C^{\alpha}_p,C^{\alpha}_{\infty}) \leq ||h||_{C^{\alpha}_p} + t\ ||g||_{C^{\alpha}_{\infty}}$$

$$\leq c\ (\int_0^{t^p} [f^* + f^{\#*}_{\alpha,p}]^p)^{1/p}.$$

This inequality can be reversed by using the subadditivity of

$$\int_0^{t^p} [(f^{\#*}_{\alpha,p})^p + (f^*)^p]. \quad \square$$

Remark: One can also characterize the K functional for the pair $(C^0_p,C^0_{\infty}) = (L_p,BMO)$, see [2].

The embedding theorems of §9 also hold when $p < 1$.

Theorem 12.5. If $0 < p \leq q \leq \infty$; $0 \leq \beta \leq \alpha + n/p - n/q$, then $C^{\alpha}_p \to C^{\beta}_q$.

Proof. This is the extension of Theorem 9.6 to $p < 1$ with essentially the same proof. To begin with, let us note that Lemma 6.6 remains valid for $p < 1$. Indeed the same argument given in the proof of this lemma shows that for any $r > 0$,

$$f^{\#}_{\beta,r}(x) \leq c\ [M_rf(x)]^{1-\theta}\ [f^{\#}_{\alpha,r}(x)]^{\theta} \leq c\ [M_rf(x) + f^{\#}_{\alpha,r}(x)]$$

with $\theta := \beta/\alpha$. We take $(\frac{1}{p} + \frac{\beta}{n})^{-1} < r < p$ and use Theorem 4.3 to find

$$(12.15) \qquad ||f^{\#}_{\beta,p}||_{L_p} \leq c\ ||f^{\#}_{\beta,r}||_{L_p} \leq c\ [||f||_{L_p} + ||f^{\#}_{\alpha,p}||_{L_p}].$$

Now suppose $\beta = \alpha + n/p - n/q$. Let P_Qf denote a best $L_q(Q)$ approximation to f of degree $[\alpha]$. From Lemma 4.4,

$$f^{\#}_{\beta,p}(x) \leq c \sup_{Q \ni x} \frac{1}{|Q|^{\beta/n}} (\frac{1}{|Q|} \int_Q |f-P_Qf|^p)^{1/p}$$

$$\leq c \sup_{Q \ni x} (|Q|^{(\alpha-\beta)/n} \inf_{u \in Q} f^{\#}_{\alpha,p}(u))$$

$$\leq c\ \{I_{\gamma}\ [(f^{\#}_{\alpha,p})^r](x)\}^{1/r}$$

with $\gamma := (\alpha-\beta)r$ and r chosen so that $0 < r < \min\ (n/(\alpha-\beta),p)$.

As in Theorem 9.3, the mapping properties of I_γ and Theorem 4.3 give

(12.16) $$|f|_{C_q^\beta} \leqq c\ ||f^\#_{\beta,p}||_{L_q} \leqq c\ |f|_{C_p^\alpha}$$

provided $q < \infty$. This inequality also holds for $q = \infty$ as can be seen from the argument in Corollary 9.4 with $f^\#_{\beta,p}$ in place of $f^\#_\beta$ and $f^\#_{\alpha,p}$ in place of $f^\#_\alpha$.

In view of (12.16), to complete the case $\beta = \alpha + n/p - n/q$ we are left with showing that $C_p^\alpha \to L_q$. For this purpose we note that Theorem 6.8 can be extended to the case $p \leqq 1$ by replacing $f_0^\#$ by $f^\#_{0,r}$ with $0 < r < p$. If $1/q_o := 1/p - \alpha/n$ is nonnegative, then it follows from (12.16) that

$$||f||_{L_{q_o}} \leqq c\ |f|_{C^o_{q_o}} \leqq c\ ||f||_{C_p^\alpha}$$

and hence $C_p^\alpha \to L_{q_o} \cap L_p \to L_q$. If $1/p - \alpha/n$ is negative, we use an analogue of Theorem 9.1. Namely, (9.2) holds with $f_\alpha^\#$ replaced by $f^\#_{\alpha,p}$ with the same proof. Arguing as in Theorem 9.6, we find

$$||f||_C \leqq c\ ||f||_{C_p^\alpha}$$

and hence $f \in C \cap L_p \subset L_q$. Thus, we have completed the case $\beta = \alpha + n/p - n/q$.

If $\beta < \alpha + n/p - n/q$, then the embedding $C_p^\alpha \to C_q^\beta$ follows from (12.15) and the case $\beta = \alpha + n/p - n/q$ proved above. □

The extension theorems of §10 and §11 hold for $p < 1$ as well. In the definition of the extension operator E for special Lipschitz domains the polynomial $P_{Q_k^s} f$ is replaced by $P_{Q_k^s} f$ a polynomial of best L_p approximation to f on Q_k^s. Again let $E_\alpha^\#$ denote the extension operator when polynomials of degree $[\alpha]$ are used and E_α^b the operator when polynomials of degree (α) are used. We then have the following analogue of Theorem 10.5.

Theorem 12.6. If Ω is a special Lipschitz domain and $p > 0$ then the extension operator $E_\alpha^\#$ is bounded from $C_p^\alpha(\Omega)$ into $C_p^\alpha(\mathbb{R}^n)$. Similarly E_α^b is bounded from $\dot{C}_p^\alpha(\Omega)$ into $\dot{C}_p^\alpha(\mathbb{R}^n)$.

Proof. In the proof, the obvious changes are made. We replace $f_\alpha^\#$ by $f_{\alpha,p}^\#$ and L_1 estimates by L_p estimates. □

We also have the analogue of Theorem 11.4.

Theorem 12.7. If Ω is a domain with minimally smooth boundary and $\alpha,p > 0$, there is an extension operator E and a constant $c > 0$ such that

$$||Ef||_{C_p^\alpha(\mathbb{R}^n)} \leqq c\ ||f||_{C_p^\alpha(\Omega)} .$$

Proof. Lemmas 11.1 and 11.2 hold for $p < 1$ with no essential change in the proof. In Lemma 11.3, we use $f_{\alpha,q}^\#$ with $(\frac{\alpha}{n} + \frac{1}{p})^{-1} < q < p$ in place of $f_\alpha^\#$ and analogous maximal functions ${}_jf_{\alpha,q}$ in place of ${}_jf_\alpha$. Also the Hardy-Littlewood maximal function M is replaced by M_q. These changes are used then in the proof of Theorem 11.4. □

Using Theorem 12.7, various results for $\mathbb{R}^n$ can be proven for domains Ω with minimally smooth boundaries. Most notably the embeddings of Theorem 12.3 follow for these Ω and it still holds that $C_p^\alpha(\Omega)$ is an interpolation space between $C_{p_o}^\alpha(\Omega)$ and $C_{p_1}^\alpha(\Omega)$ provided $0 < p_o < p < p_1 \leqq \infty$.

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