Finite Element Approximations to One-Phase Nonlinear Free Boundary Problem in Groundwater Contamination Flow

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In this article, we consider a single-phase coupled nonlinear Stefan problem of the water-head and concentration equations with nonlinear source and permeance terms and a Dirichlet boundary condition depending on the free-boundary function. The problem is very important in subsurface contaminant transport and remediation, seawater intrusion and control, and many other applications. While a Landau type transformation is introduced to immobilize the free boundary, a transformation for the water-head and concentration functions is defined to deal with the nonhomogeneous Dirichlet boundary condition, which depends on the free boundary function. An H^1-finite element method for the problem is then proposed and analyzed. The existence of the approximation solution is established, and error estimates are obtained for both the semi-discrete schemes and the fully discrete schemes. Numer Methods Partial Differential Eq 22: 1267–1288, 2006

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I. INTRODUCTION

The objective of numerical simulation for the groundwater contamination flow is to predict, control, and remediate the contamination in subsurface contaminant transport and remediation, seawater intrusion and control, and many other applications (see, for example, [1–5]). In this study, we will consider an H^1-finite element method for the one-dimensional, single-phase coupled nonlinear Stefan problem arising in groundwater contamination flow.

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Let $H(y,\tau)$ denote the water-head function, $v(y,\tau)$ denote the Darcy's velocity of groundwater, $c(y,\tau)$ be the concentration of contaminant, and $s(\tau)$ be the free boundary function. The one-dimensional mathematical model is a single-phase nonlinear Stefan problem in the following form.

Problem $(\mathcal{P})$: Find a pair $\{H(y,\tau), c(y,\tau)\}$ such that

$$S_s \frac{\partial H}{\partial \tau} - \frac{\partial}{\partial y}\left(\frac{K}{\mu}\frac{\partial H}{\partial y}\right) = f(H), \qquad \text{in } \Omega(\tau) \times (0, T_0], \tag{1.1}$$

$$v = -\frac{K}{\mu}\frac{\partial H}{\partial y}, \qquad \text{in } \Omega(\tau) \times (0, T_0], \tag{1.2}$$

$$\phi \frac{\partial c}{\partial \tau} + v\frac{\partial c}{\partial y} - \frac{\partial}{\partial y}\left(D\frac{\partial c}{\partial y}\right) = g(c), \qquad \text{in } \Omega(\tau) \times (0, T_0], \tag{1.3}$$

with the initial and boundary conditions

$$H(0,\tau) = \alpha_0, \qquad H(s(\tau),\tau) = s(\tau), \quad \tau \in (0, T_0], \tag{1.4}$$

$$c(0,\tau) = \beta_0, \qquad c(s(\tau),\tau) = \beta_1, \quad \tau \in (0, T_0], \tag{1.5}$$

$$H(y,0) = H_0(y), \qquad c(y,0) = c_0(y), \quad y \in I, \tag{1.6}$$

and on the free boundary $y = s(\tau)$, the free boundary function $s(\tau)$ satisfies

$$\frac{ds}{d\tau} = -\frac{K}{\nu\mu}\frac{\partial H}{\partial y} + \omega(H), \qquad \tau \in (0, T_0], \tag{1.7}$$

with the initial value $s(0) = 1$.

Where $S_s > 0$ is the storativity constant, $K > 0$ is the hydraulic conductivity, $\mu > 0$ is the fluid viscosity, $\phi > 0$ is the porosity of medium, $\nu > 0$ is the specific yield constant, $D > 0$ is the diffusion constant, $f(H)$ and $g(c)$ are the flow source functions, $\omega(H)$ is the permeance rate, $H_0(y)$ and $c_0(y)$ are the initial water-head and concentration functions, and α_0, β_0, and β_1 are the given water-head and concentration values. $T_0 > 0$ is the time period, $I = (0,1)$ is the initial domain, and $\Omega(\tau) = \{y : 0 < y < s(\tau)\}$, for $\tau \in (0, T_0]$, is the moving domain. The mass conservation of fluid incorporated with compressible medium, Darcy's law and the mass conservation of contaminant lead to the water-head equation (1.1) and the concentration equation (1.3). Since the unsteady free surface is a material or liquid surface composed of the same particles, the free boundary equation (1.7) characterizes that the hydrodynamic derivative for free boundary is zero from the point of view of an observer moving with particle.

Among numerical methods for free boundary problems, the front fixing methods have been successful in simulation of single-phase Stefan problems. In a series of articles, Nitsche [6, 7] proposed a finite element method for a linear free boundary problem with a Dirichlet boundary condition by straightening the free boundary and established an error analysis for a semi-discrete scheme. Das and Pani [8–11] extended the method to a single-phase nonlinear Stefan problem with zero boundary conditions and obtained optimal error estimates for both semi-discrete and fully discrete Galerkin approximations. Gastaldi in [12] considered the approach of Nitsche for a linear Stefan-like problem with a generalized free boundary condition. Based on the front fixing technique, Asaithambi in [13] studied a variable time-step approximation to a one-dimensional Stefan problem describing the evaporation of droplets. D'Ambra in [14] discussed numerical simulation of the growth of a crystal from its melt by using a front fixing method. The article [15] discussed an H^1-Galerkin method for the nonlinear Stefan system of the water-head and free boundary equations, derived the global existence of the approximate solution, and obtained optimal error estimates of the approximation.

For the groundwater contamination flow, the model is a more complex coupled system of the water-head, concentration, and free boundary equations. The free boundary equation (1.7) depends on the water-head variable H, the Dirichlet boundary condition (1.4) depends on the free boundary function $s(\tau)$, and the domain $\Omega(\tau)$ depends on the free boundary function $s(\tau)$. Due to the nonlinearity, the couplings, and the free boundary (free surface or phreatic surface), solving the system is more difficult. Therefore, there are considerable interests to develop efficient numerical methods for dealing with these features for the groundwater contamination problem in porous media. In the article, we propose and analyze an H^1-Galerkin method for the general single-phase nonlinear Stefan system of water-head, concentration, and free boundary equations [i.e., Problem $(\mathcal{P})$ (1.1)–(1.7)]. For treating the free boundary, a Landau type transformation is introduced to transform the problem into one with a fixed domain similarly as in the previous works (see, [6–15]). For overcoming the difficulty that the water-head boundary condition depends on the free boundary function $s(\tau)$, a transformation for the water-head and concentration functions is given to transform the nonhomogeneous Dirichlet condition into a normal homogeneous one. Because of the nonlinear source terms $f(H)$ and $g(c)$, the nonlinear permeance term $\omega(H)$, and the dependence of the Dirichlet boundary value on the free boundary function $s(\tau)$, the transformed problem becomes a system of two nonlinear parabolic equations and one nonlinear ordinary differential equation. We make use of the theory of variation methods, the Schauder's fixed point theorem and the technique of prior estimates to analyze this numerical procedure. The global existence of the approximate solution is proved and optimal error estimates are obtained for both semi-discrete and fully discrete finite element schemes. The theoretical analysis of the proposed method for the nonlinear coupled Stefan problem in this article is more difficult compared with that in the previous articles. Thus, the present work has significance in both theoretical analysis and application for groundwater contamination flows in porous media.

The rest of this article is organized as follows. In Section 2, a continuous time H^1-finite element scheme and basic assumptions are presented for groundwater contamination flow. Auxiliary projections and prior error estimates are given in Section 3. Error estimates for the continuous time H^1-finite element schemes are derived in Section 4. In Section 5, the global existence of the approximation solution is proved. Finally, the fully discrete finite element schemes are proposed and analyzed in Section 6.

II. CONTINUOUS TIME FINITE ELEMENT SCHEMES AND BASIC ASSUMPTIONS

In this section, we will define the continuous approximation schemes for the groundwater contamination problem. Meanwhile, some basic assumptions, notations, and properties will be introduced in order to analyze the existence and error estimates of the schemes in the following sections.

Let $W^{r,p}(\Omega)$ be normal Sobolev spaces on domain Ω for $1 \leq p \leq \infty$ and for nonnegative integer r. For $p = 2$, we simply write $H^r(\Omega)$ in place of $W^{r,p}(\Omega)$ with norm $\|\cdot\|_r$. Let $X(\tau)$ be a Banach space for each $\tau \geq 0$ with norm $\|\cdot\|_{X(\tau)}$. The following notations are used (see, [6,16], etc.):

$$\|v\|_{L^p(0,T;X(\tau))} = \left(\int_0^T \|v(\tau)\|^p_{X(\tau)}\right)^{1/p}, \qquad \text{for } 1 \leq p < \infty, \tag{2.1}$$

$$\|v\|_{L^\infty(0,T;X(\tau))} = \sup_{0\leq\tau\leq T} \|v(\tau)\|^p_{X(\tau)}, \qquad \text{for } p = \infty, \tag{2.2}$$

where $X(\tau)$ is $H^r(\Omega)$ or $H^r(I)$.

In order to overcome the difficulty of moving domain, we take the Landau-type transformation

$$x = [s(\tau)]^{-1} y, \tag{2.3}$$

and the time-scale transformation

$$t = t(\tau) = \int_0^\tau [s(\tau')]^{-2} d\tau'. \tag{2.4}$$

Using transformations (2.3) and (2.4), Problem $(\mathcal{P})$ can be transformed into a problem with the fixed domain $I \times (0, T]$, where $t = T$ corresponds to $\tau = T_0$. Because the water-head boundary condition (1.4) depends on the free boundary function $s(\tau)$, which makes it difficult to do the practical computation and theoretical analysis of numerical methods, we further introduce the following transformation:

$$p(x,t) = H(y(x), \tau(t)) - xs(\tau(t)) - (1-x)\alpha_0, \qquad (x,t) \in I \times (0,T], \tag{2.5}$$

$$b(x,t) = c(y(x), \tau(t)) - x\beta_1 - (1-x)\beta_0, \qquad (x,t) \in I \times (0,T]. \tag{2.6}$$

Let $s(\tau) = q(t)$, $p_x = \partial p/\partial x$, $b_x = \partial b/\partial x$, and $p_x(1) = (\partial p/\partial x)(1,t)$, we then obtain the following transformed problem to Problem $(\mathcal{P})$ by applying transformations (2.3)–(2.6).

Problem $(\mathcal{Q})$: Find $\{p(x,t), b(x,t), q(t)\}$ such that

$$\begin{aligned} S_s \frac{\partial p}{\partial t} - \frac{\partial}{\partial x}\left(K \frac{\partial p}{\partial x}\right) = {} & -\frac{1}{\nu\mu} S_s K p_x(1) x p_x + \varphi_1(q) p_x + \frac{1}{\nu\mu} S_s K \alpha_0 x p_x(1) \\ & + \varphi_2(q) + q^2 f(p + xq + (1-x)\alpha_0), \qquad (x,t) \in I \times (0,T], \end{aligned} \tag{2.7}$$

$$u = -\frac{K}{\mu}\frac{\partial p}{\partial x}, \qquad (x,t) \in I \times (0,T], \tag{2.8}$$

$$\begin{aligned} \phi \frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} - \frac{\partial}{\partial x}\left(D \frac{\partial b}{\partial x}\right) = {} & -\frac{1}{\nu\mu}\phi K p_x(1) x b_x - \frac{1}{\nu\mu}\phi K(\beta_1 - \beta_0) p_x(1) x \\ & + \frac{1}{\mu} K(\beta_1 - \beta_0) p_x + \psi_1(q) b_x + \psi_2(q) \\ & + q^2 g(b + x\beta_1 + (1-x)\beta_0), \qquad (x,t) \in I \times (0,T], \end{aligned} \tag{2.9}$$

$$\frac{dq}{dt} = \frac{1}{\nu\mu} K[\alpha_0 - p_x(1)] q + \varphi_0(q) q^2, \qquad t \in (0,T], \tag{2.10}$$

$$p(0,t) = p(1,t) = 0, \qquad t \in (0,T], \tag{2.11}$$

$$b(0,t) = b(1,t) = 0, \qquad t \in (0,T], \tag{2.12}$$

$$p(x,0) = H_0(x) - x - (1-x)\alpha_0, \qquad x \in I, \tag{2.13}$$

$$b(x,0) = c_0(x) - x\beta_1 - (1-x)\beta_0, \qquad x \in I, \tag{2.14}$$

$$q(0) = 1, \tag{2.15}$$

where

$$\varphi_1(q) = S_s\omega(q)qx + -\frac{1}{\nu\mu}S_sK(\alpha_0 - q)x,$$

$$\varphi_2(q) = -\frac{1}{\nu\mu}S_sK(\alpha_0 - q)x\alpha_0 - S_s\omega(q)q\alpha_0 x,$$

$$\psi_1(q) = \phi\omega(q)qx + \left(\frac{1}{\mu} - \frac{1}{\nu\mu}\phi Kx\right)(q - \alpha_0),$$

$$\psi_2(q) = \frac{2}{\mu}K(q-\alpha_0)(\beta_1 - \beta_0) - \frac{1}{\nu\mu}\varphi K(q-\alpha_0)x(\beta_1-\beta_0) - \phi\omega(q)qx(\beta_1-\beta_0),$$

$$\varphi_0(q) = \omega(q) - \frac{1}{\nu\mu}K.$$

Remark 2.1. *Problem* ($\mathcal{Q}$) *is a system in a fixed domain of a nonlinear parabolic water-head equation* (2.7)*, a nonlinear parabolic concentration equation* (2.9)*, and a nonlinear ordinary differential free boundary equation* (2.10)*. After solving p, b, and q, the original solutions H, c, and s of Problem* ($\mathcal{P}$) *can be obtained as follows:*

$$H(y,\tau) = p(x,t) + xq(t) + (1-x)\alpha_0, \tag{2.16}$$

$$c(y,\tau) = b(x,t) + x\beta_1 + (1-x)\beta_0, \tag{2.17}$$

$$s(\tau) = q(t), \tag{2.18}$$

where $y = qx$ and $\tau = \tau(t)$ satisfy equation

$$\frac{d\tau}{dt} = (q(t))^2, \qquad 0 < t \le T, \tag{2.19}$$

with the initial value $\tau(0) = 0$.

Remark 2.2. *In the groundwater flow, $y = s(\tau)$ is the free surface or phreatic surface, which satisfies in general that $0 < s_0 \le s(\tau), \tau \in (0, T_0]$ for finite time period $T_0 > 0$. If it degenerates to $s(T_0) = 0$, there would be no flow any more in the subsurface and we will not deal with this degenerated case here. Actually, with proper source and permeance terms and a certain initial distribution, the groundwater flow naturally satisfies that $0 < s_0 \le s(\tau), \tau \in (0, T_0]$. Some theoretical results for the solution variables and $s(\tau)$ have been obtained for similar Stefan problems (see, for example,* [17–21]*). In this article, we will provide condition $0 < s_0 \le s(\tau), \tau \in (0, T_0]$ for a given finite time period $T_0 > 0$. With this condition, the Landau-type transformation* (2.3) *was used for some free boundary problems in the previous articles* (*see* [6–15]*, etc*).

In order to carry out our theoretical analysis for Problem ($\mathcal{P}$) and Problem ($\mathcal{Q}$), we make, throughout this article, the following assumptions, which we call Assumption (A). *Assumption* (A):

(i) Problem ($\mathcal{P}$) has a unique smooth solution $\{H, c, s\}$ for all $y \in \Omega(\tau)$ and $\tau \in (0, T_0]$ satisfying that $0 < s_0 \le s(\tau), \tau \in (0, T_0]$, and

$$H \in W^{1,2}(0, T_0; H^{l+1}(\Omega(\tau))) \cap W^{1,\infty}(0, T_0; H^2(\Omega(\tau))), \qquad s \in W^{1,\infty}(0, T_0),$$
$$c \in W^{1,2}(0, T_0; H^{m+1}(\Omega(\tau))) \cap W^{1,\infty}(0, T_0; H^2(\Omega(\tau))).$$

(ii) $f, g, \omega \in C^1(\mathbf{R})$ with locally uniformly bounded derivatives with bound M_1.
(iii) The initial function H_0 is sufficiently smooth and satisfies the compatibility conditions $H_0(0) = 0$ and $H_0(1) = 1$.

We further assume that the uniqueness and regularity properties for $\{H, c, s\}$ can be carried over to the solution $\{p, b, q\}$ of Problem ($\mathcal{Q}$), and that

$$p \in W^{1,2}(0,T; H^{l+1}(I)) \cap W^{1,\infty}(0,T; H^2(I)), \qquad q \in W^{1,\infty}(0,T), \tag{2.20}$$

$$b \in W^{1,2}(0,T; H^{m+1}(I)) \cap W^{1,\infty}(0,T; H^2(I)). \tag{2.21}$$

Remark 2.3. *The nonlinear functions f, g, and ω, and their derivatives are required only to be locally uniformly bounded, which allows a wide class of problems to be included in our results. Regarding the existence, uniqueness, and regularity results of the solution, the reader is referred to* [17–21]*, etc.*

In order to derive a weak formulation for Problem ($\mathcal{Q}$), let $H_0^1(I) = \{v \in H^1(I) \mid v(0) = v(1) = 0\}$. Multiplying both sides of (2.7) by $-v_{xx}$ for $v \in H^2(I) \cap H_0^1(I)$ and integrating by parts the first term with respect to x, we obtain that for $t \in (0, T]$

$$\begin{aligned}(S_s p_{tx}, v_x) + ((Kp_x)_x, v_{xx}) = \frac{1}{\nu\mu} S_s K p_x(1)(x p_x, v_{xx}) - (\varphi_1(q) p_x, v_{xx}) - (\varphi_2(q), v_{xx}) \\ - \left(\frac{1}{\nu\mu} S_s K \alpha_0 x p_x(1), v_{xx}\right) - (q^2 f(p + xq + (1-x)\alpha_0), v_{xx}).\end{aligned} \tag{2.22}$$

Similarly, we get from (2.9) by multiplying $-w_{xx}$ for $w \in H^2(I) \cap H_0^1(I)$ that for $t \in (0, T]$

$$\begin{aligned}(\phi b_{tx}, w_x) - (u b_x, w_{xx}) + ((D b_x)_x, w_{xx}) = \frac{1}{\nu\mu} \phi K p_x(1)(x b_x, w_{xx}) - (\psi_1(q) b_x, w_{xx}) \\ - \left(\frac{1}{\mu} K(\beta_1 - \beta_0) p_x, w_{xx}\right) - \left(\frac{1}{\nu\mu} \phi K(\beta_1 - \beta_0) p_x(1) x, w_{xx}\right) \\ - (\psi_2(q), w_{xx}) - (q^2 g(b + x\beta_1 + (1-x)\beta_0), w_{xx}).\end{aligned} \tag{2.23}$$

The weak formulation for Problem ($\mathcal{Q}$) is obtained as follows: Find $p(t) \in H^2(I) \cap H_0^1(I)$, $b(t) \in H^2(I) \cap H_0^1(I)$, and $q(t)$ satisfying (2.22) for all $v \in H^2(I) \cap H_0^1(I)$, (2.23) for all $w \in H^2(I) \cap H_0^1(I)$, and (2.10) as well as initial conditions (2.13), (2.14), and (2.15).

We now propose a continuous time H^1-finite element approximate scheme for Problem ($\mathcal{Q}$). Let $0 < h_p < 1$ and $0 < h_b < 1$ be the spatial step size for pressure and the spatial step size for concentration, respectively. Let $V_{h_p} \subset H^2(I) \cap H_0^1(I)$ be C^1-finite element space with index $l \geq 1$ and $W_{h_b} \subset H^2(I) \cap H_0^1(I)$ be C^1-finite element space with index $m \geq 1$, which satisfy the following properties (see [7] and [16]):

(i) Approximation property: for $2 \leq r \leq l + 1$ and $2 \leq k \leq m + 1$, it holds that for some constants C_1 and C_2

$$\inf_{v_h \in V_{h_p}} \|v - v_h\|_j \leq C_1 h_p^{r-j} \|v\|_r, \qquad \text{for } j = 0, 1, 2;\ v \in H^r(I) \cap H_0^1(I), \tag{2.24}$$

$$\inf_{w_h \in W_{h_b}} \|v - w_h\|_j \leq C_1 h_b^{k-j} \|w\|_k, \qquad \text{for } j = 0, 1, 2;\ w \in H^k(I) \cap H_0^1(I). \tag{2.25}$$

(ii) Inverse property: for $v_h \in V_{h_p}$, $w_h \in W_{h_b}$ and some constants C_3 and C_4 independent of v_h and w_h, it holds that

$$\|v_h\|_2 \le C_3 h_p^{-1} \|v_h\|_1, \tag{2.26}$$

$$\|w_h\|_2 \le C_4 h_b^{-1} \|w_h\|_1. \tag{2.27}$$

The continuous time finite element scheme for Problem $(\mathcal{Q})$ is defined as follows: Find $p^h(t) \in V_{h_p}$, $b^h(t) \in W_{h_b}$, and q^h such that for $t \in (0, T]$

$$(S_s p^h_{tx}, v_x) + ((K p^h_x)_x, v_{xx}) = \frac{1}{\nu\mu} S_s K p^h_x(1)(x p^h_x, v_{xx}) - (\varphi_1(q^h) p^h_x, v_{xx}) - (\varphi_2(q^h), v_{xx})$$
$$- \left(\frac{1}{\nu\mu} S_s K \alpha_0 x p^h_x(1), v_{xx}\right) - ((q^h)^2 f(p^h + x q^h + (1-x)\alpha_0), v_{xx}), v(t) \in V_{h_p}, \tag{2.28}$$

$$(\phi b^h_{tx}, w_x) - (u^h b^h_x, w_{xx}) + ((D b^h_x)_x, w_{xx}) = \frac{1}{\nu\mu} \phi K p^h_x(1)(x b^h_x, w_{xx})$$
$$- (\psi_1(q^h) b^h_x, w_{xx}) - \left(\frac{1}{\mu} K(\beta_1 - \beta_0) p^h_x, w_{xx}\right) - \left(\frac{1}{\nu\mu} \phi K(\beta_1 - \beta_0) p^h_x(1) x, w_{xx}\right)$$
$$- (\psi_2(q^h), w_{xx}) - ((q^h)^2 g(b^h + x\beta_1 + (1-x)\beta_0), w_{xx}), \quad w(t) \in W_{h_b}, \tag{2.29}$$

$$\frac{dq^h}{dt} = \frac{1}{\nu\mu} K[\beta - p^h_x(1)] q^h + \varphi_0(q^h)(q^h)^2, \tag{2.30}$$

with $u^h = -(1/\mu) K p^h_x$, and the initial values $q^h(0) = 1$, $p^h(x,0) = \Theta_{h_p}(H_0(x) - x - (1-x)\alpha_0)$, and $b^h(x,0) = \Theta_{h_b}(c_0(x) - x\beta_1 - (1-x)\beta_0)$, where Θ_{h_p} and Θ_{h_b} are appropriate projection operators onto V_{h_p} and W_{h_b} to be defined later.

Remark 2.4. *Once we get p^h, b^h, and q^h by solving* (2.28)–(2.30) *for Problem* $(\mathcal{Q})$*, an approximate solution $\{H^h, c^h, s^h\}$ to Problem $(\mathcal{P})$ can be obtained as follows:*

$$H^h(y,\tau) = p^h(x,t) + x q^h(t) + (1-x)\alpha_0, \tag{2.31}$$

$$c^h(y,\tau) = b^h(x,t) + x\beta_1 + (1-x)\beta_0, \tag{2.32}$$

$$s^h(\tau) = q^h(t), \tag{2.33}$$

where $y = q^h x$ and $\tau = \tau^h(t)$ satisfy the equation

$$\frac{d\tau^h(t)}{dt} = (q^h(t))^2, \qquad t > 0, \tag{2.34}$$

with initial value $\tau^h(0) = 0$.

III. AUXILIARY PROJECTIONS AND PRIOR ESTIMATES

For establishing the existence of the approximation solutions and analyzing the errors of the approximation schemes, respectively, we will define the auxiliary elliptic projections associated with solution $\{p, b, q\}$ and estimate the related errors of the projections.

For $p, v, w \in H^2(I)$, define

$$\mathcal{A}_\rho(p, q; v, w) = \left(\left(\frac{K}{\mu} v_x\right)_x, w_{xx}\right) - \frac{1}{\nu\mu} S_s K p_x(1)(x v_x, w_{xx}) + (\varphi_1(q) v_x, w_{xx}) + \rho_1(v_x, w_x), \quad (3.1)$$

$$\mathcal{B}_\rho(p, q; v, w) = ((D v_x)_x, w_{xx}) + (u v_x, w_{xx}) - \frac{1}{\nu\mu} \phi K p_x(1)(x v_x, w_{xx}) + (\psi_1(q) v_x, w_{xx}) + \rho_2(v_x, w_x), \quad (3.2)$$

where $u = -(1/\mu) K p_x$, and $\rho_1 > 0$ and $\rho_2 > 0$ are sufficiently large constants so that there are constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\mathcal{A}_\rho(p, q; v, v) \geq \gamma_1 \|v\|_2^2, \qquad \forall v \in H^2(I) \cap H_0^1(I), \quad (3.3)$$

$$\mathcal{B}_\rho(p, q; w, w) \geq \gamma_2 \|w\|_2^2, \qquad \forall w \in H^2(I) \cap H_0^1(I). \quad (3.4)$$

In addition, we can easily show that $\mathcal{A}_\rho(p, q; v, w)$ and $\mathcal{B}_\rho(p, q; v, w)$ are bounded in $H^2(I) \cap H_0^1(I)$, that is,

$$|\mathcal{A}_\rho(p, q; v, w)| \leq M_2 \|v\|_2 \|w\|_2, \qquad \forall v, w \in H^2(I) \cap H_0^1(I), \quad (3.5)$$

$$|\mathcal{B}_\rho(p, q; v, w)| \leq M_3 \|v\|_2 \|w\|_2, \qquad \forall v, w \in H^2(I) \cap H_0^1(I), \quad (3.6)$$

for some positive constants M_2 and M_3, depending only on $\|p\|_2$, $\|b\|_1$, and q.

For $t \in (0, T]$, the auxiliary elliptic projections of $p(x, t)$ and $b(x, t)$ onto V_{h_p} with respect to $\mathcal{A}_\rho$ and onto W_{h_b} with respect to $\mathcal{B}_\rho$ are defined as follows: Find $\tilde{p}(x, t) \in V_{h_p}$ and $\tilde{b}(x, t) \in W_{h_b}$ such that

$$\mathcal{A}_\rho(p, q; p - \tilde{p}, v) = 0, \qquad \forall v \in V_{h_p}, \quad (3.7)$$

$$\mathcal{B}_\rho(p, q; b - \tilde{b}, w) = 0, \qquad \forall w \in W_{h_b}. \quad (3.8)$$

Noting the properties (3.3)–(3.6), the existence of unique $\tilde{p}(x, t)$ and $\tilde{b}(x, t)$ in (3.7) and (3.8) follows from the Lax-Milgram theorem. Moreover, the prior error estimates for $\eta = p - \tilde{p}$ and $\theta = b - \tilde{b}$, and a superconvergence estimate for $\eta_x(1, t)$ in Lemma 3.1 and Lemma 3.2 can be obtained similarly as in [8–10].

Lemma 3.1. *For $t \in (0, T]$, the errors $\eta = p - \tilde{p}$ and $\theta = b - \tilde{b}$ satisfy*

$$\|\eta\|_j \leq C h_p^{r-j} \|p\|_r, \qquad \|\eta_t\|_j \leq C h_p^{r-j} (\|p_t\|_r + \|p\|_r), \quad (3.9)$$

$$\|\theta\|_j \leq C h_b^{k-j} \|p\|_k, \qquad \|\theta_t\|_j \leq C h_b^{k-j} (\|p_t\|_k + \|p\|_k), \quad (3.10)$$

for $j = 0, 1, 2$, and $2 \leq r \leq l + 1$, $2 \leq k \leq m + 1$, where C is a positive constant independent of h_p and h_b.

Lemma 3.2. *There exists a constant C such that*

$$|\eta_x(1, t)| \leq C h_p^{2(r-2)} \|p\|_r, \qquad 2 \leq r \leq l + 1. \quad (3.11)$$

Proof. Let $\chi \in H^4(I) \cap H_0^1(I)$ be a solution of the following elliptic problem:

$$\mathcal{L}^*(p,q)\chi_{xx} = 0, \quad x \in I, \qquad \chi_{xx}(0) = 0, \qquad \chi_{xx}(1) = 1, \tag{3.12}$$

where $\mathcal{L}^*(p,q)$ is the elliptic operator defined by

$$\mathcal{L}^*(p,q)\chi = \frac{\partial}{\partial x}\left(K\frac{\partial \chi}{\partial x}\right) + \frac{1}{\mu\nu}Kp_x(1)\frac{\partial}{\partial x}(x\chi) - \frac{\partial}{\partial x}(\varphi_1(q)\chi) - \rho_1\chi.$$

Multiplying both sides of Equation (3.12) by η, and integrating by parts with respect to x, we get from Equation (3.7) and relations (3.5) (2.24) that

$$\begin{aligned}|K\eta_x(1,t)| = |\mathcal{A}_\rho(p,q;\eta,\chi)| &= \inf_{v^h \in V_h} |\mathcal{A}_\rho(p,q;\eta,\chi - v^h)| \\ &\le M_2\|\eta\|_2 \inf_{v^h \in V_{h_p}} \|\chi - v^h\|_2 \le C_1 M_2 h_p^{r-2}\|\eta\|_2\|\chi\|_r. \end{aligned} \tag{3.13}$$

By the elliptic regularity, $\|\chi\|_m$ is bounded above with a bound depending only on q and p. The required estimate (3.11) then follows from (3.13) and (3.9) for $j = 2$. ∎

IV. ERROR ESTIMATES FOR CONTINUOUS TIME SCHEMES

In this part we will focus on error estimates of the continuous time finite element scheme (2.28)–(2.30). We first assume that the finite element approximation $\{p^h, b^h, q^h\}$ exists, then we will prove the existence and uniqueness of solution in next section. Let $e_p = p - p^h$, $\xi = \tilde{p} - p^h$, $\eta = p - \tilde{p}$, $e_b = b - b^h$, $\pi = \tilde{b} - b^h$, $\theta = b - \tilde{p}$, and $e_q = q - q^h$, we have that $e_p = \xi + \eta$, $e_b = \pi + \theta$. For simplicity, we choose the initial approximations $\Theta_{h_p} p(x,0) = \tilde{p}(x,0)$ and $\Theta_{h_b} b(x,0) = \tilde{b}(x,0)$ in (2.28)–(2.30), where $\tilde{p}(x,0)$ and $\tilde{b}(x,0)$ are the elliptic projections of $p(x,0)$ and $b(x,0)$ onto V_{h_p} and W_{h_b} defined in (3.7) and (3.8), respectively. Thus, $\xi(x,0) = 0$ and $\pi(x,0) = 0$.

For obtaining error estimates conveniently, we assume that there exist two positive constants M^* and $0 < h_0 < 1$ independent of h_p and h_b, such that, for $0 < h_p, h_b \le h_0$,

$$\|p^h\|_{L^\infty(0,T;H^2(I))} + \|b^h\|_{L^\infty(0,T;H^2(I))} + \|q^h\|_{L^\infty(0,T)} \le M^*. \tag{4.1}$$

The condition (4.1) is actually true, which will be proved in Theorem 4.1.

Lemma 4.1. *Suppose that Assumption* (*A*) *and* (2.20), (2.21) *hold. Let* $\{p, b, q\}$ *be the solution of Problem* ($\mathcal{Q}$), *and let* $\{p^h, b^h, q^h\}$ *be the finite element approximation defined in* (2.28)–(2.30). *Assume that condition* (4.1) *holds. Then there are two positive constants* $0 < h_0 < 1$ *and* C *independent of* h_p *and* h_b, *such that, for* $l \ge 2$, $m \ge 2$ *and* $0 < h_p, h_b \le h_0$,

$$\begin{aligned}&\|\xi_x\|^2(t) + \|\pi_x\|^2(t) + \|e_q\|^2(t) + \|\xi_{xx}\|^2_{L^2(0,T;L^2(I))} + \|\pi_{xx}\|^2_{L^2(0,T;L^2(I))} \\ &\quad \le C\big(h_p^{2l}(\|p\|^2_{L^2(0,T;H^{l+1}(I))} + \|p_t\|^2_{L^2(0,T;H^{l+1}(I))}) + h_b^{2m}(\|b\|^2_{L^2(0,T;H^{m+1}(I))} + \|b_t\|^2_{L^2(0,T;H^{m+1}(I))})\big),\end{aligned} \tag{4.2}$$

for $0 < t \le T$.

Proof. First, we consider the free-boundary error equation. It follows from (2.10) and (2.30) that

$$\frac{d\pi}{dt} = \frac{1}{\nu\mu}K[p_x^h(1)q^h - p_x(1)q] + \frac{1}{\nu\mu}\alpha_0 K(q - q^h) + [\varphi_0(q)q^2 - \varphi_0(q^h)(q^h)^2] \equiv G_1 + G_2 + G_3. \quad (4.3)$$

Using Assumption (A), (2.20), (4.1), and the definition of $\varphi_0(q)$, we can obtain the estimates

$$|G_1| \le C_0(|\eta_x(1)| + |\xi_x(1)| + |e_q|), \qquad |G_2 + G_3| \le C_0|e_q|,$$

for constant $C_0 = C_0(M^*)$.

Multiplying (4.3) by e_q and making use of Cauchy-Schwarz's inequality, we get that for arbitrary $\epsilon > 0$,

$$\frac{d}{dt}|e_q|^2 \le C_\epsilon(M^*)(|\eta_x(1)|^2 + |e_q|^2) + \epsilon\|\xi_{xx}\|_0^2. \quad (4.4)$$

Next, we turn to the derivation of a corresponding evolution inequality for ξ. Using (2.22), (2.28), (3.7), and (3.1), we get the following water-head error equation:

$$\begin{aligned}(S_s\xi_{tx}, v_x) + A_\rho(p, q; \xi, v) &= -(S_s\eta_{tx}, v_x) + \rho_1((e_p)_x, v_x) \\ &+ \frac{1}{\mu\nu}S_s K((p_x(1) - p_x^h(1))xp_x^h, v_{xx}) + \frac{1}{\nu\mu}S_s K\alpha_0((p_x^h(1) - p_x(1))x, v_{xx}) \\ &+ ((\varphi_1(q^h) - \varphi_1(q))p_x^h, v_{xx}) + (\varphi_2(q^h) - \varphi_2(q), v_x x)) \\ &+ ((q^h)^2 f(p^h + xq^h + (1-x)\alpha_0) - q^2 f(p + xq + (1-x)\alpha_0), v_{xx}) \\ &\equiv G_4 + G_5 + \cdots + G_{10}, \quad (4.5)\end{aligned}$$

for $v \in V_{h_p}$. We take $v = \xi$ in (4.5) and estimate each term on both sides of (4.5). Along with the fact that $\eta_t(0) = \eta_t(1) = e(0) = e(1) = 0$, integrating by parts leads to that

$$|G_4 + G_5| \le (S_s\|\eta_t\|_0 + \rho_1\|e\|_0)\|\xi_{xx}\|_0. \quad (4.6)$$

Since $\xi \in V_h \subset H_0^1(I)$, the inequality $\|\xi_x\|_{L^\infty(I)} \le \sqrt{2}\|\xi\|_0^{1/2}\|\xi_{xx}\|_0^{1/2}$ (Ladyzhenskaya et al. [22]) and Poincare's inequality, it implies that

$$|\xi_x(1)| \le \sqrt{2}\|\xi_x\|_0^{1/2}\|\xi_{xx}\|_0^{1/2}. \quad (4.7)$$

Applying (4.7) to G_6 and G_7, we can get that

$$|G_6 + G_7| \le \sqrt{2}\frac{1}{\mu\nu}S_s K(C_1(M^*) + \alpha_0)(|\eta_x(1)|\|\xi_{xx}\|_0 + |\xi_x(1)|^{1/2}\|\xi_{xx}\|_0^{3/2}), \quad (4.8)$$

where $C_1(M^*)$ independent of h_p and h_b. Similarly, we estimate G_8, G_9, and G_{10}. Substituting the estimates of G_j $(j = 4, \ldots, 10)$ into (4.5), and using (3.3), we obtain the inequality

$$\frac{S_s}{2}\frac{d}{dt}\|\xi_x\|_0^2 + \gamma_1\|\xi_{xx}\|_0^2 \le C_\epsilon[\|\eta_t\|_0^2 + \|\eta\|_0^2 + |\eta_x(1)|^2 + (\|\xi_x\|_0^2 + |e_q|^2)] + \epsilon\|\xi_{xx}\|_0^2, \quad (4.9)$$

for arbitrary $\epsilon > 0$.

Finally, from (2.23), (2.29), (3.2), and (3.8), we obtain the concentration error equation as follows:

$$
\begin{aligned}
(\phi\pi_{tx}, w_x) + B_\rho(p,q;\pi,w) &= -(\phi\theta_{tx}, w_x) + \rho_2((e_b)_x, w_x) + (ub_x^h - u^h b_x^h, w_{xx}) \\
&\quad + \frac{1}{\mu\nu}\phi K((p_x(1) - p_x^h(1))xb_x^h, w_{xx}) + \frac{1}{\nu\mu}\phi K(\beta_1 - \beta_0)((p_x^h(1) - p_x(1))x, w_{xx}) \\
&\quad + \frac{1}{\mu}K(\beta_1 - \beta_0)(p_x^h - p_x, w_{xx}) + ((\psi_1(q^h) - \psi_1(q))b_x^h, w_{xx}) + (\psi_2(q^h) - \psi_2(q), w_{xx}) \\
&\quad + ((q^h)^2 g(b^h + x\beta_1 + (1-x)\beta_0) - q^2 g(b + x\beta_1 + (1-x)\beta_0), w_{xx}) \\
&\equiv G_{11} + G_{12} + \cdots + G_{19}. \qquad (4.10)
\end{aligned}
$$

Setting $w = \pi$ in (4.10), we estimate the terms on the right-hand side of (4.10). Noting the definitions of u and u^h, we get by using condition (4.1) that

$$
|G_{13}| \le \left|\left(-\frac{K}{\mu}(p_x - p_x^h)b_x^h, w_{xx}\right)\right| \le C_2(M^*)\frac{K}{\mu}(\|\eta_x\|_0 + \|\xi_x\|_0)\|\pi_{xx}\|_0, \qquad (4.11)
$$

and

$$
\begin{aligned}
|G_{19}| &\le |((q^h)^2 - q^2)g(b^h + x\beta_1 + (1-x)\beta_0) \\
&\quad + q^2(g(b^h + x\beta_1 + (1-x)\beta_0) - g(b + x\beta_1 + (1-x)\beta_0)), w_{xx})| \\
&\le C_3(M^*)(|e_q|_0 + \|e_b\|_0)\|\pi_{xx}\|_0, \qquad (4.12)
\end{aligned}
$$

where $C_2(M^*)$ and $C_3(M^*)$ are independent of h_p and h_b.

Other G's terms are estimated similarly. Notice that $e_p = \eta + \xi$, and $e_b = \theta + \pi$, we obtain that

$$
\begin{aligned}
\frac{\phi}{2}\frac{d}{dt}\|\pi_x\|_0^2 + \gamma_2\|\pi_{xx}\|_0^2 &\le C_\epsilon[\|\theta_t\|_0^2 + \|\theta\|_0^2 + \|\eta_x\|_0^2 + +|\eta_x(1)|^2 \\
&\quad + (\|\xi_x\|_0^2 + \|\pi_x\|_0^2 + |e_q|^2)] + \epsilon(\|\xi_{xx}\|_0^2 + \|\pi_{xx}\|_0^2). \qquad (4.13)
\end{aligned}
$$

Taking ϵ sufficiently small in (4.4), (4.9), and (4.13) yields

$$
\begin{aligned}
&\frac{d}{dt}(\|\xi_x\|_0^2 + \|\pi_x\|_0^2 + |e_q|^2) + (\|\xi_{xx}\|_0^2 + \|\pi_{xx}\|_0^2) \\
&\quad \le C_6(M^*)[\|\eta_t\|_0^2 + \|\eta\|_0^2 + \|\eta_x\|_0^2 + \|\theta_t\|_0^2 + \|\theta\|_0^2 + |\eta_x(1)|^2 + (\|\xi_x\|_0^2 + \|\pi_x\|^2 + |e_q|^2)].
\end{aligned}
$$

An application of the Gronwall's inequality to the above inequality gives

$$
\begin{aligned}
\|\xi_x(t)\|_0^2 + \|\pi(t)\|_0^2 + |e_q(t)|^2 + \int_0^t (\|\xi_{xx}(\tau)\|_0^2 + \|\pi_{xx}(\tau)\|_0^2)d\tau &\le C_5(M^*, T)\int_0^t [\|\eta_t(\tau)\|_0^2 \\
&\quad + \|\eta(\tau)\|_0^2 + \|\eta_x(\tau)\|_0^2 + |\eta_x(1,\tau)|^2 + \|\theta_t(\tau)\|_0^2 + \|\theta(\tau)\|_0^2]d\tau.
\end{aligned}
$$

The required estimate (4.2) with C depending on M^* under condition (4.1) is obtained from (4.13) by using Lemma 3.1 and Lemma 3.2. This ends the proof. ∎

Theorem 4.2. *Suppose that Assumption* (A) *and* (2.20), (2.21) *hold. Let* $\{p, b, q\}$ *be the solution of Problem* $(\mathcal{Q})$, *and let* $\{p^h, b^h, q^h\}$ *be the finite element approximation in* (2.28)–(2.30). *There exist constants* $0 < h_0 < 1$ *and* $C > 0$ *independent of* h_p *and* h_b, *such that, for* $0 < h_p, h_b \leq h_0, l \geq 2$ *and* $m \geq 2$, *and*

$$h_p^{-1} h_b^m \to 0, \qquad h_b^{-1} h_p^l \to 0, \qquad for\ h_p, h_b \to 0, \tag{4.14}$$

it holds that

$$\|p^h\|_{L^\infty(0,T;H^2(I))} + \|b^h\|_{L^\infty(0,T;H^2(I))} + \|q^h\|_{L^\infty(0,T)} \leq M^*, \tag{4.15}$$

and the following error estimates:

$$\|p - p^h\|_{L^\infty(0,T;H^1(I))} + h_p\|p - p^h\|_{L^\infty(0,T;H^2(I))} \leq C(h_p^l + h_b^m), \tag{4.16}$$

$$\|b - b^h\|_{L^\infty(0,T;H^1(I))} + h_b\|b - b^h\|_{L^\infty(0,T;H^2(I))} \leq C(h_p^l + h_b^m), \tag{4.17}$$

$$\|q - q^h\|_{L^\infty(0,T)} \leq C(h_p^l + h_b^m). \tag{4.18}$$

Proof. If (4.15) is true, the error estimates can be easily derived from Lemma 4.1, Lemma 3.1, and Lemma 3.2 together with the approximation properties (2.24), (2.25), and the inverse properties (2.26), (2.27). So we only need to show that (4.15) is actually true.

Let

$$M_0 = \|p\|_{L^\infty(0,T;H^2(I))} + \|b\|_{L^\infty(0,T;H^2(I))} + \|q\|_{L^\infty(0,T)},$$

then, we may assume without loss of generality that $M^* \geq 3M_0$. Noting that $p^h(x,0) = \tilde{p}(x,0)$, $b^h(x,0) = \tilde{b}(x,0)$, and $q^h(0) = 1$, we have from Lemma 3.1 that

$$\|p^h(0) - p(0)\|_2 + \|b^h(0) - b(0)\|_2 + |q^h(0)| = \|\tilde{p}(0) - p(0)\|_2 \\ + \|\tilde{b}(0) - b(0)\|_2 + 1 \leq Ch_p^{l-1}\|p(0)\|_{l+1} + Ch_b^{m-1}\|b(0)\|_{m+1} + 1,$$

for constant $C > 0$, so that

$$\|p^h(0)\|_2 + \|b^h(0)\|_2 + |q^h(0)| \leq \|p(0)\|_2 + \|b(0)\|_2 \\ + Ch_p^{l-1}\|p(0)\|_{l+1} + Ch_b^{m-1}\|b(0)\|_{m+1} + 1 \leq 2M_0, \tag{4.19}$$

for sufficiently small h_p and h_b. If (4.15) were false, by (4.19) and the continuity in t of $\|p^h(t)\|_2 + \|b^h(t)\|_2 + |q^h(t)|$, there would be a t_1 independent of h_p and h_b such that $t_1 < T$ and

$$\|p^h(t)\|_2 + \|b^h(t)\|_2 + |q^h(t)| \leq M^*, \qquad t \in [0, t_1), \tag{4.20}$$

$$\|p^h(t_1)\|_2 + \|b^h(t_1)\|_2 + |q^h(t_1)| > M^*, \tag{4.21}$$

for sufficiently small h_p and h_b. Since $p - p^h = \eta + \xi$ and $b - b^h = \theta + \pi$, using (4.2) to ξ and π for $t \in (0, t_1]$, Lemma 3.1 to η and θ, and the inverse properties (2.26), (2.27), we obtain that for $l \geq 2$ and $m \geq 2$,

$$h_p\|p - p^h\|_{L^\infty(0,t_1;H^2(I))} + h_b\|b - b^h\|_{L^\infty(0,t_1;H^2(I))} + \|q - q^h\|_{L^\infty(0,t_1)} \\ \leq Ch_p^l(\|p\|_{L^2(0,T;H^{l+1}(I))} + \|p_t\|_{L^2(0,T;H^{l+1}(I))}) \\ + Ch_b^m(\|b\|_{L^2(0,T;H^{m+1}(I))} + \|b_t\|_{L^2(0,T;H^{m+1}(I))}) \equiv Ch_p^l F_1 + Ch_b^m F_2.$$

So, for sufficiently small h_p and h_b,

$$\begin{aligned}\|p^h(t_1)\|_2 + \|b^h(t_1)\|_2 + |q^h(t_1)| &\le \|p(t_1)\|_2 + \|b(t_1)\| + |q(t_1)| \\ &\quad + C[(h_p^{l-1} + h_b^{-1}h_p^l)F_1 + (h_b^{m-1} + h_p^{-1}h_b^m)F_2] \\ &\le M_0 + C[(h_p^{l-1} + h_b^{-1}h_p^l)F_1 + (h_b^{m-1} + h_p^{-1}h_b^m)F_2].\end{aligned}$$

Taking h_p and h_b sufficiently small such that

$$C[(h_p^{l-1} + h_b^{-1}h_p^m)F_1 + (h_b^{m-1} + h_p^{-1}h_b^l)F_2] < M_0,$$

we have

$$\|p^h(t_1)\|_2 + \|b^h(t_1)\|_2 + |q^h(t_1)| < 2M_0 < M^*,$$

which contradicts (4.21). Here use has been made of the facts $l \ge 2, m \ge 2$, and (4.14). The proof is thus complete. ∎

Further, noting formulas (2.16)–(2.18), definitions (2.31)–(2.33), and relations (2.19) and (2.34), and using Theorem 4.1, we can easily obtain the following error estimates for the finite element approximation $\{H^h, c^h, s^h\}$ of the solution $\{H, c, s\}$ to Problem $(\mathcal{P})$.

Theorem 4.3. *Suppose that Assumption (A) is satisfied. Let $\{H, c, s\}$ be the solution of Problem (P). Let $\{H^h, c^h, s^h\}$ be the finite element approximation defined through $\{p^h, b^h, q^h\}$ in* (2.31)–(2.33). *Then, if $l \ge 2, m \ge 2$, and* (4.14) *holds, we have the following error estimates:*

$$\|s - s^h\|_{L^\infty(0,T_0)} + \|\tau - \tau^h\|_{L^\infty(0,T_0)} = O(h_p^l + h_b^m), \tag{4.22}$$

$$\|H - H^h\|_{L^\infty(0,T_0;H^1(\tilde{\Omega}(\tau)))} + h_p\|H - H^h\|_{L^\infty(0,T_0;H^2(\tilde{\Omega}(\tau)))} = O(h_p^l + h_b^m), \tag{4.23}$$

$$\|c - c^h\|_{L^\infty(0,T_0;H^1(\tilde{\Omega}(\tau)))} + h_b\|c - c^h\|_{L^\infty(0,T_0;H^2(\tilde{\Omega}(\tau)))} = O(h_p^l + h_b^m), \tag{4.24}$$

where $\tilde{\Omega}(\tau) = (0, \min[s(\tau), s^h(\tau)])$ for $\tau \in (0, T_0]$.

V. GLOBAL EXISTENCE OF THE APPROXIMATE SOLUTION

Now, we consider existence and uniqueness of the finite element approximation $\{p^h, b^h, q^h\}$ by employing Schauder's fixed point theorem. To do this, we recall the system of equations (4.3), (4.5), and (4.10), which is a system of nonlinear ordinary differential equations in ξ, π, and e_q. For using Schauder's fixed point theorem we need to linearize equations (4.3), (4.5), and (4.10).

In Equation (4.5), noting $e_p = \eta + \xi$ and the fact that for all $v \in V_{h_p}$

$$\rho(e_x, v_x) = -\rho(\eta + \xi, v_{xx}), \qquad -(\mu\eta_{tx}, v_x) = (\mu\eta_t, v_{xx}),$$

and replacing $p^h = p - e_p$ in terms G_6 and G_7, we get that

$$G_6 = \frac{1}{\mu\nu} S_s K((\eta_x(1) + \xi_x(1))x(p - e_p)_x, v_{xx}), \tag{5.1}$$

$$G_7 = \frac{1}{\mu\nu} S_s K\alpha_0(-(\eta_x(1) + \xi_x(1))x, v_{xx}). \tag{5.2}$$

And since

$$\varphi_1(q^h) - \varphi_1(q) = -(q - q^h)\int_0^1 \varphi_1'(q - t'(q - q^h))dt', \tag{5.3}$$

we thus have for G_8 that

$$G_8 = \left(-(q - q^h)\int_0^1 \varphi_1'(q - t'(q - q^h))dt'(p_x - (e_p)_x), v_{xx}\right). \tag{5.4}$$

Similarly, treat G_9 and G_{10}. Substituting e_p by $E_p(x,t)$ and e_q by $E_q(t)$ for some functions $E_p \in L^\infty(0,T; H^2(I) \cap H_0^1(I))$ and $E_q \in L^\infty(0,T)$, we obtain a linearized formulation for Equation (4.5):

$$\begin{aligned}
(S_s\xi_{tx}, v_x) + \mathcal{A}_\rho(p,q;\xi,v) &= (S_s\eta_t, v_{xx}) + \rho_1(\eta + \xi, v_{xx}) \\
&+ \frac{1}{\mu\nu}S_sK((\eta_x(1) + \xi_x(1))x(p_x - (E_p)_x), v_{xx}) + \frac{1}{\nu\mu}S_sK\alpha_0(-(\eta_x(1) + \xi_x(1))x, v_x x) \\
&+ \left(-e_q\int_0^1 \varphi_1'(q - t'E_q)dt'(p_x - (E_p)_x), v_{xx}\right) + \left(-e_q\int_0^1 \varphi_2'(q - t'E_q)dt', v_{xx}\right) \\
&\quad + (e_q(E_q - 2q)f((p - E_p) + xE_q + (1 - x)\alpha_0) \\
&\quad - q^2(\eta + \xi - e_q)\int_0^1 f'(p + xq + (1 - x)\alpha_0 - t'(E_p + xE_q))dt', v_{xx}).
\end{aligned} \tag{5.5}$$

To treat terms G_j $(j = 11, \ldots, 19)$ in Equation (4.10) in the same way as above by using $e_p = \eta + \xi$, $e_b = \theta + \pi$, $p^h = p - e_p$, $b^h = b - e_b$, and $q^h = q - e_q$, and substituting e_p by $E_p(x,t)$, e_q by $E_q(t)$, and e_b by $E_b(x,t)$, it turns out for the linearized concentration formulation that

$$\begin{aligned}
(\phi\pi_{tx}, w_x) + B_\rho(p,q;\pi,v) &= (\phi\theta_t, w_{xx}) + \rho_2(\theta + \pi, w_{xx}) \\
&+ \left(-\frac{K}{\mu}(\eta_x + \xi_x)(b_x - (E_b)_x), w_{xx}\right) + \frac{1}{\mu\nu}\phi K((\eta_x(1) + \xi_x(1))x(b_x - (E_b)_x), w_{xx}) \\
&+ \frac{1}{\nu\mu}\phi K(\beta_1 - \beta_0)(-(\eta_x(1) + \xi_x(1))x, w_{xx}) + \frac{1}{\mu}K(\beta_1 - \beta_0)(-(\eta_x + \xi_x), w_{xx}) \\
&+ \left(-e_q\int_0^1 \psi_1'(q - E_qt')dt'\right)(b - E_b)_x, w_{xx}) + \left(-e_q\int_0^1 \psi_2'(q - E_qt'), w_{xx}\right) \\
&\quad + (e_q(E_q - 2q)g((b - E_b) + x\beta_1 + (1 - x)\beta_0) \\
&\quad - q^2(\theta + \pi)\int_0^1 g'(b + x\beta_1 + (1 - x)\beta_0 - t'E_b)dt', w_{xx}).
\end{aligned} \tag{5.6}$$

Similarly, from (4.3) we obtain the linearized free-boundary equation:

$$\begin{aligned}
\frac{de_q}{dt} &= \frac{1}{\nu\mu}K(-(\eta_x(1) + \xi_x(1))(q - E_q) - p_x(1)e_q) + \frac{1}{\nu\mu}\alpha_0 Ke_q \\
&\quad + e_q\varphi_0(q - E_q)(2q - E_q) + e_qq^2\int_0^1 \varphi_0'(q - t'E_q)dt'.
\end{aligned} \tag{5.7}$$

Then, rearranging the right-hand side of (5.5), (5.6), and (5.7), we obtain a coupled system of three linear ordinary differential equations in ξ, π, and e_q for giving $E_p \in L^\infty(0,T;H^2(I)\cap H_0^1(I))$, $E_b \in L^\infty(0,T;H^2(I)\cap H_0^1(I))$, and $E_q \in L^\infty(0,T)$,

$$(S_s\xi_{tx}, v_x) + \mathcal{A}_\rho(p,q;\xi,v) = (\chi_1(E_p,E_q)\xi + \chi_2(E_p)\xi_x(1) + \chi_3(E_p,E_q)e_q + \chi_4(E_p,E_q)\eta + \chi_5(E_p)\eta_x(1) + S_s\eta_t, v_{xx}), \qquad \forall v \in V_{h_p}, \quad (5.8)$$

$$(\phi\pi_{tx}, w_x) + \mathcal{B}_\rho(p,q;\pi,w) = (\chi_6(E_b)\pi + \chi_7(E_b)\xi_x + \chi_8(E_b 1)\xi_x(1) + \chi_9(E_b,E_q)e_q + \chi_{10}(E_b)\theta + \chi_{11}(E_b)\eta_x + \chi_{12}(E_b)\eta_x(1) + \phi\theta_t, w_{xx}), \qquad \forall w \in W_{h_b}, \quad (5.9)$$

$$\frac{de_q}{dt} = \chi_{13}(E_q)e_q + \chi_{14}(E_q)\xi_x(1) + \chi_{15}(E_q)\eta_x(1), \quad (5.10)$$

with initial values $\xi(0) = 0$, $\pi(0) = 0$, and $e_q(0) = 0$, where $\chi_i(E_p,E_q)$, $\chi_i(E_b,E_q)$, $\chi_i(E_q)$, $\chi_i(E_p)$, and $\chi_i(E_b)$ are functions independent of h, ξ, and π. Since the system of Equations (5.8), (5.9), and (5.10) is a linear ordinary differential system in ξ, π, and e_q as functions of t for any given functions $E_p = E_p(x,t)$, $E_b = E_b(x,t)$, and $E_q = E_q(t)$, there exists a unique solution $\{\xi, \pi, e_q\}$ of system (5.8), (5.9), and (5.10) in the interval $(0,T]$.

Let U and U_h represent the spaces:

$$U := L^\infty(0,T;H^2(I)\cap H_0^1(I)) \times L^\infty(0,T;H^2(I)\cap H_0^1(I)) \times L^\infty(0,T), \quad (5.11)$$
$$U_h := L^\infty(0,T;V_{h_p}) \times L^\infty(0,T;W_{h_b}) \times L^\infty(0,T). \quad (5.12)$$

Thus, the linear system of Equations (5.8), (5.9), and (5.10) defines an operator $\mathcal{T}$ from U into U_h such that

$$\{\xi, \pi, e_q\} = \mathcal{T}\{E_p, E_b, E_q\}, \quad (5.13)$$

for each $\{E_p, E_b, E_q\} \in U$. Since $e_p = \eta + \xi, e_b = \eta + \xi$, then $\{e_p, e_b, e_q\} = \{\eta, \theta, 0\} + \mathcal{T}\{E_p, E_b, E_q\}$ for each $\{E_p, E_b, E_q\} \in U$. Now, define operator $\mathcal{D} : U \to U$ by

$$\mathcal{D}\{E_p, E_b, E_q\} = \{\eta, \theta, 0\} + \mathcal{T}\{E_p, E_b, E_q\}. \quad (5.14)$$

Thus, by employing Schauder's fixed point theorem, to show the existence of a solution $\{p^h, b^h, q^h\}$ to problem (2.28)–(2.30), we only need to show that operator $\mathcal{D}$ has a fixed point $\{E_p, E_b, E_q\}$ in U, that is,

$$\mathcal{D}\{E_p, E_b, E_q\} = \{E_p, E_b, E_q\}. \quad (5.15)$$

Theorem 5.1. *Suppose that Assumption* (*A*) *and* (2.20), (2.21) *are satisfied. Let* $\{p,b,q\}$ *be the solution of Problem* (*Q*). *If* (4.14) *holds and* $0 < \delta \le 1, l \ge 2$ *and* $m \ge 2$, *then for sufficiently small* h_p *and* h_b, *there exists a unique finite element solution* $\{p^h, b^h, q^h\} \in L^\infty(0,T;V_{h_p}) \times L^\infty(0,T;W_{h_b}) \times L^\infty(0,T)$ *to problem* (2.28)–(2.30) *such that*

$$\|p - p^h\|_{L^\infty(0,T;H^2(I))} + \|b - b^h\|_{L^\infty(0,T;H^2(I))} + \|q - q^h\|_{L^\infty(0,T)} \le \delta. \quad (5.16)$$

Proof. First, we consider Equation (5.8). Let $v = \xi$ in (5.8) and use (3.3) and (3.5) to obtain that

$$\frac{S_s}{2}\frac{d}{dt}\|\xi_x\|_0^2 + \gamma_1\|\xi\|_2^2 \le \|\xi_{xx}\|_0\{S_s\|\eta_t\|_0 + C[(1+|q|^2)(\|\eta\|_0 + \|\xi\|_0) + (1 + \|p_x\|_0 + \|(E_p)_x\|_0)(|\eta_x(1)| + |\xi_x(1)|) + (1 + |q| + |q|^2 + \|p_x\|_0 + |E_q| + \|(E_p)_x\|_0)|e_q|)]\|\xi_{xx}\|,$$

for positive constant C. Applying Cauchy-Schwarz's inequality to the above inequality, we obtain that for arbitrary $\epsilon > 0$,

$$\frac{S_s}{2}\frac{d}{dt}\|\xi_x\|_0^2 + \gamma_1\|\xi\|_2^2 \le \epsilon\|\xi_{xx}\|^2 + C_\epsilon\{\|\eta_t\|_0^2 + \|\eta\|_0^2 \\ + (1 + \|(E_p)_x\|_0^2)|\eta_x(1)|^2 + (1 + \|(E_p)_x\|_0^4 + |E_q|^2)(\|\xi_x\|_0^2 + |e_q|^2)]\}. \quad (5.17)$$

Second, choosing $w = \pi$ in (5.9) and making use of (3.4) and (3.6), we get that

$$\frac{\phi}{2}\frac{d}{dt}\|\pi_x\|_0^2 + \gamma_2\|\pi\|_2^2 \le \|\pi_{xx}\|_0\{\phi\|\theta_t\|_0 + C[(1 + |q|^2)(\|\theta\|_0 + \|\pi\|_0) \\ + (1 + \|b_x\|_{L^\infty(I)} + \|(E_b)_x\|_{L^\infty(I)})(\|\eta_x\|_0 + \|\xi_x\|_0) + (1 + \|b_x\|_0 \\ + \|(E_b)_x\|_0)(|\eta_x(1)| + |\xi_x(1)|) + (1 + |q| + \|b_x\|_0 + |E_q| + \|(E_b)_x\|_0)|e_q|)]\|\pi_{xx}\|.$$

It yields that for arbitrary $\epsilon > 0$,

$$\frac{\phi}{2}\frac{d}{dt}\|\pi_x\|_0^2 + \gamma_2\|\pi\|_2^2 \le \epsilon(\|\pi_{xx}\|_0^2 + \|\xi_{xx}\|_0^2) + C_\epsilon\{\|\theta_t\|_0^2 + \|\theta\|_0^2 \\ + (1 + \|(E_b)_x\|_{L^\infty(I)}^2)\|\eta_x\|^2 + (1 + \|(E_b)_x\|_0^2)|\eta_x(1)|^2 \\ + (1 + \|(E_b)_x\|_0^4 + \|(E_b)_x\|_{L^\infty(I)}^2 + |(E_q)|^2)(\|\pi_x\|_0^2 + \|\xi_x\|_0^2 + |e_q|^2)\}. \quad (5.18)$$

Finally, multiplying (5.10) by e_q and applying Cauchy-Schwarz's inequality give

$$\frac{1}{2}\frac{d|e_q|^2}{dt} \le \epsilon\|\xi_{xx}\|^2 + C_\epsilon(1 + |E_q|^2)(|e_q|^2 + |\eta_x(1)|^2). \quad (5.19)$$

Now, combining (5.17)–(5.19) and taking ϵ sufficiently small, we have that

$$\frac{d}{dt}(\|\xi_x\|_0^2 + \|\pi_x\|_0^2 + |e_q|^2) \le C_2\{\|\eta_t\|_0^2 + \|\theta_t\|_0^2 + \|\eta\|_0^2 + \|\theta\|_0^2 \\ + (1 + \|(E_b)_x\|_{L^\infty(I)}^2)\|\eta_x\|_0^2 + (1 + \|(E_p)_x\|_0^2 + \|(E_b)_x\|_0^2 + |E_q|^2)|\eta_x(1)|^2\} \\ + C_2(1 + |E_q|^2 + \|(E_b)_x\|_{L^\infty(I)}^2 + \|(E_p)_x\|_0^4 + \|(E_b)_x\|_0^4)(\|\xi_x\|_0^2 + \|\pi_x\|^2 + |e_q|^2), \quad (5.20)$$

for constant $C_2 > 0$ independent of h_p, h_b, E_p, E_b, and E_q. Taking the supremum over $(0, T]$ with respect to t and making use of Lemma 3.1–3.2, we obtain by applying the Gronwall's inequality to (5.20) and the inverse properties (2.26) and (2.27) that

$$h_p^2\|\xi\|_{L^\infty(0,T;H^2(I))}^2 + h_b^2\|e_q\|_{L^\infty(0,T;H^2(I))}^2 + \|\pi\|_{L^\infty(0,T)}^2 \le C_4(h_p^{2l} + h_b^{2m}), \quad (5.21)$$

with some $C_4 > 0$ independent of h_p and h_b but depending on E_p, E_b, and E_q.

Now, we introduce the notation

$$|\,\|\{v, w, q\}\|\,| := \|v\|_{L^\infty(0,T;H^2(I))} + \|w\|_{L^\infty(0,T;H^2(I))} + \|q\|_{L^\infty(0,T)},$$

for $\{v, w, q\} \in U$. Let $\{E_p, E_b, E_q\}$ satisfy

$$|||\{E_p, E_b, E_q\}||| \le \delta \le 1.$$

Then, using Lemma 3.1 and the definition of $\mathcal{D}$, and noting that $l \ge 2$ and $m \ge 2$, we get from (5.21) that for $0 < h_p, h_b \le 1$,

$$\begin{aligned} |||\mathcal{D}\{E_p, E_b, E_q\}||| \le \|\eta\|_{L^\infty(0,T;H^2(I))} + \|\theta\|_{L^\infty(0,T;H^2(I))} + \|\xi\|_{L^\infty(0,T;H^2(I))} \\ + \|\pi\|_{L^\infty(0,T;H^2(I))} + \|e_q\|_{L^\infty(0,T)} \le C_5(h_p^{l-1} + h_b^{-1}h_p^m + h_b^{l-1} + h_b^{-1}h_p^l), \end{aligned}$$

where $C_5 > 0$ is independent of h_p, h_b, E_p, E_b, and E_q. Noting (4.14), there thus exists an $0 < h_0 < 1$ such that for all $0 < h_p, h_b \le h_0$, it holds that $|||\mathcal{D}\{E_p, E_b, E_q\}||| \le \delta$. So, operator $\mathcal{D}$ maps the sphere

$$B_\delta := \{\{E_p, E_b, E_q\} \in U \ : \ |||\{E_p, E_b, E_q\}||| \le \delta\}, \qquad (0 < \delta \le 1)$$

into itself for $0 < h_p, h_b \le h_0$. Clearly, $\mathcal{D}$ is continuous and compact. Therefore, by applying the Schauder's fixed point theorem, there exits a solution $\{E_p, E_b, E_q\}$ in B_δ such that $\mathcal{D}\{E_p, E_b, E_q\} = \{E_p, E_b, E_q\}$. The proof of the existence of the approximation solution is complete. The uniqueness of the solution can be easily obtained from the theory of ordinary differential equations. ∎

VI. FULLY DISCRETE FINITE ELEMENT SCHEMES

In this section, we first define a fully discrete finite element scheme for Problem ($\mathcal{Q}$) as well as a fully discrete approximation to Problem ($\mathcal{P}$). We then analyze the error estimates for the fully discrete schemes. Finally, we describe briefly how to implement the schemes and how to extend the methods to multidimensional problems at the end of this section.

Let $\Delta t > 0, N = T/\Delta t \in Z$, and $t^n = n\Delta t, n = 0, 1, \ldots, N$. Let $p^n = p(x, t^n), b^n = b(x, t^n)$, and $d_t p^n = (p^n - p^{n-1})/\Delta t$, $d_t b^n = (b^n - b^{n-1})/\Delta t$. Denote the approximation of p^n by P^n in V_{h_p}, the approximation of b^n by B^n in W_{h_b}, and the approximation of q^n by Q^n. Assuming that P^{n-1}, B^{n-1}, and Q^{n-1} are known, we determine P^n, B^n, and Q^n by the following fully discrete finite element scheme: Find $P^n \in V_{h_p}$, $B^n \in W_{h_p}$, and Q^n such that

$$\begin{aligned} (S_s d_t P_x^n, v_x) + ((K P_x^n)_x, v_{xx}) = \frac{1}{\nu\mu} S_s K P_x^{n-1}(1)(x P_x^n, v_{xx}) \\ - (\varphi_1(Q^{n-1})P_x^n, v_{xx}) - (\varphi_2(Q^{n-1}), v_{xx}) - \left(\frac{1}{\nu\mu} S_s K \alpha_0 x P_x^n(1), v_{xx}\right) \\ - ((Q^{n-1})^2 f(P^{n-1} + xQ^{n-1} + (1-x)\alpha_0), v_{xx}), \qquad v \in V_{h_p}, \end{aligned} \quad (6.1)$$

$$d_t Q^n = \frac{1}{\nu\mu} K[\alpha_0 - P_x^n(1)]Q^n + \varphi_0(Q_{n-1})(Q^{n-1})^2, \quad (6.2)$$

$$\begin{aligned} (\phi d_t B_x^n, w_x) - (U^n B_x^n, w_{xx}) + ((DB_x^n)_x, w_{xx}) = \frac{1}{\nu\mu}\phi K P_x^n(1)(x B_x^n, w_{xx}) \\ - (\psi_1(Q^n)B_x^{n-1}, w_{xx}) - \left(\frac{1}{\mu} K(\beta_1 - \beta_0)P_x^n, w_{xx}\right) - \left(\frac{1}{\nu\mu}\phi K(\beta_1 - \beta_0)P_x^n(1)x, w_{xx}\right) \\ - (\psi_2(Q^n), w_{xx}) - ((Q^n)^2 g(B^{n-1} + x\beta_1 + (1-x)\beta_0), w_{xx}), \qquad w \in W_{h_b}, \end{aligned} \quad (6.3)$$

for $n = 1, 2, \ldots, N$, and with $U^n = -(1/\mu)KP_x^n$, the initial values $P^0 = \Theta_{h_p} p(x, 0)$, $B^0 = \Theta_{h_b} b(x, 0)$ and $Q^0 = 1$, where Θ_{h_p} and Θ_{h_b} are appropriate projection operators onto V_{h_p} and W_{h_b}, respectively. Since scheme (6.1)–(6.3) leads to a system of linear algebraic equations for P^n, B^n, and Q^n, which has a solution for sufficiently small Δt for any given initial values $\{P^0, B^0, Q^0\}$. In the rest of this section, we shall derive error estimates for the fully discrete scheme (6.1)–(6.3). In order to do this, we make the following additional smoothness assumptions for the solutions $\{H, c, s\}$ and $\{p, b, q\}$, and functions f and ω, which we call *Assumption* (B).

Assumption (B): In addition to Assumption (A), we assume further that $\{H, c, s\}$ and $\{p, b, q\}$ satisfy

$$H \in W^{1,2}(0, T_0; H^{l+1}(\Omega(\tau))) \cap W^{2,2}(0, T_0; H^2(\Omega(\tau))), \qquad s \in W^{2,2}(0, T_0), \tag{6.4}$$

$$c \in W^{1,2}(0, T_0; H^{m+1}(\Omega(\tau))) \cap W^{2,2}(0, T_0; H^2(\Omega(\tau))), \tag{6.5}$$

and

$$p \in W^{1,2}(0, T; H^{l+1}(I)) \cap W^{2,2}(0, T; H^2(I))), \qquad q \in W^{2,2}(0, T), \tag{6.6}$$

$$b \in W^{1,2}(0, T; H^{m+1}(I)) \cap W^{2,2}(0, T; H^2(I))), \tag{6.7}$$

and $f, \omega \in C^2(\mathbf{R})$ with locally uniformly bounded derivatives.

Let $e_p^n = p^n - P^n, \eta^n = p^n - \tilde{p}^n, \xi^n = \tilde{p}^n - P^n, e_b^n = b^n - B^n, \theta^n = b^n - \tilde{B}^n, \pi^n = \tilde{b}^n - B^n$, and $e_q^n = q^n - Q^n$, from (2.22), (3.7), and (6.1), it follows that for $v \in V_{h_p}$,

$$\begin{aligned}(S_s d_t \xi_x^n, v_x) + \mathcal{A}_\rho(p^n, q^n; \xi^n, v) &= -(S_s d_t \eta + S_s[p_t^n - d_t p^n], v_{xx}) + \rho_1((e_p)_x^n, v_x) \\ &\quad + \frac{1}{\mu\nu} S_s K((p_x^n(1) - P_x^{n-1}(1))xP_x^n, v_{xx}) + \frac{1}{\nu\mu} S_s K\alpha_0((P_x^n(1) - p_x^n(1))x, v_x x) \\ &\quad + ((\varphi_1(Q^{n-1})P_x^{n-1} - \varphi_1(q^n)p_x^n, v_{xx}) + (\varphi_2(Q^{n-1}) - \varphi^2(q^n), v_{xx})) \\ &\quad + ((Q^{n-1})^2 f(P^{n-1} + xQ^{n-1} + (1-x)\alpha_0) - (q^n)^2 f(p^n + xq^n + (1-x)\alpha_0), v_{xx}),\end{aligned} \tag{6.8}$$

and for $w \in W_{h_b}$,

$$\begin{aligned}(\phi d_t \pi_x^n, w_x) + \mathcal{B}_\rho(p^n, q^n; \pi^n, w) &= -(\phi d_t \theta_x^n + \phi[b_t^n - d_t b^n], w_{xx}) + \rho_2((e_b)_x^n, w_x) \\ &\quad + ((u^n - U^n)B_x^n, w_{xx}) + \frac{1}{\mu\nu}\phi K((p_x^n(1) - P_x^n(1))xB_x^n, w_{xx}) \\ &\quad + \frac{1}{\nu\mu}\phi K(\beta_1 - \beta_0)((P_x^n(1) - p_x^n(1))x, w_{xx}) + \frac{1}{\mu} K(\beta_1 - \beta_0)(P_x^n - p_x^n, w_{xx}) \\ &\quad + ((\psi_1(Q^n) - \psi_1(q^n))B_x^{n-1}, w_{xx}) + (\psi_2(Q^n) - \psi_2(q^n), w_{xx})) \\ &\quad + ((Q^n)^2 g(B^n + x\beta_1 + (1-x)\beta_0) - (q^n)^2 g(p^n + x\beta_1 + (1-x)\beta_0), w_{xx}),\end{aligned} \tag{6.9}$$

and

$$d_t e_q^n = \frac{1}{\nu} K[P_x^n(1)Q^n - p_x^n(1)q^n] + [d_t q^n - q_t^n] + [\varphi(q^n)(q^n)^2 - \varphi(Q^{n-1})(Q^{n-1})^2]. \tag{6.10}$$

Then, we have the following results.

Lemma 6.1. *Suppose that Assumption* (B) *holds. Let* $\{p, b, q\}$ *be the solution of Problem* $(\mathcal{Q})$, *and let* $\{P^n, B^n, Q^n\}$ *be the solution of the fully discrete scheme* (6.1)–(6.3) *with initial values* $P^0 = \tilde{p}(x, 0)$, $B^0 = \tilde{b}(x, 0)$, *and* $Q^0 = 1$. *Then, there exist constants* $0 < h_0 < 1$, $0 < \Delta_0 < 1$, *and* $C > 0$ *independent of* h_p, h_b *and* Δt, *for* $0 < h_p, h_b \leq h_0$, $0 < \Delta t \leq \Delta_0$, $l \geq 2, m \geq 2$, $\Delta t = o(h_p, h_b)$, *and*

$$h_p^{-1} h_b^m + h_b^{-1} h_p^l \to 0, \qquad for\ h_p, h_b \to 0, \tag{6.11}$$

such that

$$\|\xi_x^n\|_0^2 + \|\pi_x^n\|_0^2 + |e_q^n|^2 + \sum_{k=0}^{n} (\|\xi^k\|_2^2 + \|\pi^k\|_2^2)\Delta t \leq C[(\Delta t)^2 + h_p^{2l} + h_b^{2m}], \tag{6.12}$$

for $n = 0, 1, \ldots, N$.

Proof. Make an induction hypothesis similar to the one in Section 4. Let

$$M_0 = \max_{n=0}^{N} (\|p^n\|_2 + \|b^n\|_2 + |q^n|),$$

and assume that there are constants $0 < h_0 < 1$, $0 < \Delta_0 < 1$ and M^* ($M^* > 3M_0$) such that for $0 < h_p, h_q \leq h_0$, $0 < \Delta t \leq \Delta_0$,

$$\max_{n=0}^{N} (\|P^n\|_2 + \|B^n\|_2 + |Q^n|) \leq M^*. \tag{6.13}$$

Choosing $v = \xi^n$ and $w = \pi^n$ in (6.8) and (6.9), and multiplying (6.10) by e_q^n we get by similar arguments as in deriving (4.4)–(4.9) and the definition of (3.1) and (3.2) that, for arbitrary $\epsilon > 0$,

$$\begin{aligned}\frac{1}{\Delta t}(\|\xi_x^n\|_0^2 - \|\xi_x^{n-1}\|_0^2) + \gamma_1 \|\xi_{xx}^n\|_0^2 &\leq \epsilon(\|\xi_{xx}^n\|_0^2 + \|\xi_{xx}^{n-1}\|_0^2)\\ &\quad + C_\epsilon[(\Delta t)^2 + \|d_t \eta^n\|_0^2 + \|\eta^n\|_0^2 + \|\eta^{n-1}\|_0^2 + \|\eta_x^{n-1}\|_0^2 + |\eta_x^n(1)|^2\\ &\quad + |\eta_x^{n-1}(1)|^2 + (\|\xi_x^n\|_0^2 + \|\xi_x^{n-1}\|_0^2 + |e_q^n|^2 + |e_q^{n-1}|^2)],\end{aligned} \tag{6.14}$$

$$\begin{aligned}\frac{1}{\Delta t}(\|\pi_x^n\|_0^2 - \|\pi_x^{n-1}\|_0^2) + \gamma_2 \|\pi_{xx}^n\|_0^2 &\leq \epsilon(\|\pi_{xx}^n\|_0^2 + \|\xi_{xx}^n\|_0^2)\\ &\quad + C_\epsilon[(\Delta t)^2 + \|d_t \theta^n\|_0^2 + \|\theta^n\|_0^2 + \|\theta^{n-1}\|_0^2 + \|\eta_x^n\|_0^2 + |\eta_x^n(1)|^2\\ &\quad + (\|\pi_x^n\|_0^2 + \|\pi_x^{n-1}\|_0^2 + \|\xi_x^n\|_0^2 + |e_q^n|^2],\end{aligned} \tag{6.15}$$

$$\frac{1}{\Delta t}(|e_q^n|^2 - |e_q^{n-1}|^2) \leq \epsilon \|\xi_{xx}^n\|_0^2 + C_\epsilon(|\eta_x^n(1)|^2 + (\Delta t)^2 + \|\xi_x^n\|_0^2 + |e_q^n|^2 + |e_q^{n-1}|^2), \tag{6.16}$$

where C_ϵ is a positive constant independent of h_p, h_q, and Δt. Multiplying both sides of (6.14)–(6.16) by Δt, combining these three inequalities together, summing in time $1 \leq n \leq N$ and choosing ϵ sufficiently small, we are able to obtain the required estimate (6.12) by applying the discrete Gronwall's inequality, Lemma 3.1 and Lemma 3.2. It remains to check the induction hypothesis (6.13). Using (6.12) and a similar argument as in the proof of Theorem 4.2, it can be easily shown that (6.13) actually holds. Thus, the proof is complete. ∎

Further, from Lemma 6.1 and Lemma 3.1 together with the inverse properties (2.26) and (2.27), we easily get the following theorem.

Theorem 6.2. *Suppose that the conditions of Lemma* 6.1 *are satisfied. Then, there exist constants* $0 < h_0 < 1$, $0 < \Delta_0 < 1$ *and* $C > 0$ *independent of* h_p, h_b, *and* Δt, *for* $0 < h_p, h_q \leq h_0$, $0 < \Delta t \leq \Delta_0$, *and* $\Delta t = o(h_p, h_b)$, (6.11) *with* $l \geq 2, m \geq 2$, *such that*

$$\max_{n=0}^{N}(\|p^n - P^n\|_1^2 + \|b^n - B^n\|_1^2 + |q^n - Q^n|^2) + \sum_{n=0}^{N}(\|p^n - P^n\|_2^2 h_p + \|b^n - B^n\|_2^2 h_b)\Delta t$$
$$\leq C[(\Delta t)^2 + h_p^{2l} + h_b^{2m}]. \quad (6.17)$$

Finally, we define the fully discrete finite element approximation $\{H_h^n, c_h^n, s_h^n\}$ to the original Problem $(\mathcal{P})$ as follows:

$$H_h^n := H_h(y^n, \tau_h^n) = P^n(x) + xQ^n + (1-x)\beta, \qquad s_h^n := s_h(\tau_h^n) = Q^n, \quad (6.18)$$
$$c_h^n := c_h(y^n, \tau_h^n) = B^n(x) + x\beta_1 + (1-x)\beta_0, \quad (6.19)$$

where $y^n = xQ^n$ and τ_h^n is given via

$$d_t \tau_h^n = (Q^n)^2, \quad (6.20)$$

with initial value $\tau_h^0 = 0$. Then, from formulas (2.5), (2.6), and (2.19), and the definitions (6.18)–(6.20), and utilizing Theorem 6.1, one can easily obtain the following error estimates for the fully discrete finite element approximation $\{H_h^n, c_h^n, S_h^n\}$ of the solution $\{H, c, s\}$ to the original Problem $(\mathcal{P})$.

Theorem 6.3. *Suppose that the conditions of Theorem* 6.1 *are satisfied. Let* $\{H, c, s\}$ *be the solution of Problem* $(\mathcal{P})$, *Let* $\{H_h^n, c_h^n, S_h^n\}$ *be the fully discrete finite element approximation to Problem* $(\mathcal{P})$. *Assume that* $\Delta t = o(h_p, h_q)$ *and* (6.11) *hold. Then, we have that for* $l \geq 2$ *and* $m \geq 2$,

$$\max_{n=0}^{N}(\|H^n - H_h^n\|_{H^1(\tilde{\Omega}^n)} + \|c^n - c_h^n\|_{H^1(\tilde{\Omega}^n)}) \leq C[\Delta t + h_p^l + h_b^m], \quad (6.21)$$
$$\max_{n=0}^{N}(|s^n - s_h^n| + |\tau^n - \tau_h^n|) \leq C[\Delta t + h_p^l + h_b^m], \quad (6.22)$$

where $H^n = H(y, \tau^n)$ *and* $s^n = s(\tau^n)$ *with* τ^n *given by* (2.19), $\tilde{\Omega}^n = (0, \min[s^n, S_h^n])$, *and* $C > 0$ *is independent of* h_p, h_b, *and* Δt.

Remark 6.1. *The analysis results in the article express that the front fixing finite element method has not only the error estimates for both pressure and concentration with* $O(\Delta t + h_p^l + h_b^m)$ *but also the same order estimates for the free boundary. This guarantees that the proposed method can obtain very good approximation to the fluid flow with free surface in porous media. The condition* $\Delta t = o(h_p, h_q)$ *in Lemma* 6.1 *and Theorem* 6.1–6.2 *is provided only to obtain the high order estimate of error* $O(\Delta t + h_p^l + h_b^m)$ *since the inverse property of finite element spaces is used to derive the boundness of the approximation solutions in high-order* H^2*-norm for treating a nonlinear term in analysis. Since the indices of two* H^1*-finite element spaces are* $l \geq 2$ *and* $m \geq 2$, *it would be natural to choose* $\Delta t = o(h_p, h_q)$ *in order to reach the high accuracy of* $O(h_p^l + h_b^m)$ *from the estimates.*

Remark 6.2. *The fully discrete scheme* (6.1)–(6.3) *is a linearization iteration scheme of* $\{P^n, Q^n, B^n\}$. *The procedure can be described as follow: Step 1. Input the initial values* P^0, Q^0

and B^0*, which are the approximations of* $p(x,0)$*,* $s(0)=1$*, and* $b(x,0)$*. Step 2. For* $n=1,2,\ldots,N$ *do: (a) find* P^n *using* (6.1)*, (b) find* Q^n *using* (6.2)*, (c) find* B^n *using* (6.3)*. At steps (a) and (c), we only need to solve a triangular linear system of equations by using the Crout's factorization algorithm. Moreover, Step (b) is only a two-step scheme, which can be easily solved. From Theorem* 6.1 *and Theorem* 6.2*, we knew that the fully discrete scheme is of first order in* Δt*. We may construct some implicit schemes to increase the accuracy with respect to the time step size* Δt*, but they will increase the computational complexity of finding* P^n*,* Q^n*, and* B^n*. Once we obtain* $\{P^n, Q^n, B^n\}$*, we can easily get* $\{H^n, s_h^n, c_h^n\}$ *by using* (6.18) *and* (6.19).

Remark 6.3. *The proposed method can be extended to two-dimensional (and multidimensional) contamination flows. One extension of the method is to split the two-dimensional problem into two one-dimensional problems in the* x*-direction and the* y*-direction using the splitting technique. The* y*-directional univariate problem is a free boundary problem similar to the above one but the other in the* x*-direction is a normal one-dimensional partial differential equation. The front fixing finite element method can be used to solve the one-dimensional free boundary problem in the* y*-direction at each time step as well as the other can be used by using the standard finite element methods in the* x*-direction at each time step. The alternative iterating scheme can be used to simulate the two-dimensional free boundary problem in the subsurface. Article* [23] *has used this technique to successfully solve the subsurface flow problem with free boundary in two dimensions. Since the free boundary is only on the top surface (free surface) of flow domain, the front fixing finite element method joint with the splitting technique can efficiently simulate high-dimensional free-surface fluid flows in porous media. A rigorous analysis will be performed in the future work. Some numerical experiments related to the front fixing methods were also taken in* [13] *and* [14].

Remark 6.4. *On the other aspect, since the surface of fluid flow is only on the top surface of moving fluid, it means that the free boundary curve for the two-dimensional problem can be expressed as* $y=s(x,t)$ *at fixed time* t*. We can also expect to directly introduce the front fixing method for solving this kind of free boundary problem of fluid flow in two dimensions (or in multidimensions). Some additional analyses will be discussed in the further study.*

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