A Priori Estimates and Regularization for a Class of Porous Medium Equations [1]

by Koffi B. Fadimba and Robert C. Sharpley

Abstract

The general class of porous medium equations:

$$\frac{\partial S}{\partial t} + \nabla \cdot f(S)\mathbf{u} - \nabla \cdot k(S)\nabla S = Q(S) \ ,$$

with diffusion coefficient k vanishing for two values of saturation S and the fractional flow f having a characteristic 'S'-shaped signature, is an accepted and physically reasonable model for describing two phase flow (formulated in terms of total velocity and pressure) of groundwater in certain common soils.

For numerical studies , the possible roughness of the solutions requires an additional approximation or regularization of the equation accomplished by perturbing the diffusion coefficient k, in order to obtain a nondegenerate problem with smooth solutions. We establish a-priori estimates for solutions and quantitative estimates for the regularization of these possibly degenerate equations. It is shown that the regularized solution, S_β, converges to S as the perturbation parameter β tends to 0 with specific convergence rates specified throughout all estimates.

1. Introduction

In modeling immiscible two phase flow through a porous medium (see Peaceman [8]), the following class of saturation problems arise:

$$\frac{\partial S}{\partial t} + \nabla \cdot (f(S)\mathbf{u}) - \nabla \cdot (k(S)\nabla S) = Q(S) \quad \text{ on } \Omega \times (0, T_0] \tag{1.1a}$$

with boundary conditions:

$$(f(S)\mathbf{u} - k(S)\nabla S) \cdot \mathbf{n} = q \quad \text{ on } \partial\Omega \times [0, T_0] \tag{1.1b}$$

and initial condition:

$$S(x, 0) = S^0(x) \qquad \text{ on } \Omega \tag{1.1c}$$

where Ω is a bounded domain of $I\!R^n$, $n = 1, 2, 3$, and $0 \le S^0(x) \le 1$, for all $x \in \Omega$.

The saturation S ($0 \le S \le 1$) of the invading fluid is the ratio of the volume of the fluid (water for instance) to the volume of voids in a representative elementary volume (see Bear-Verruijt [2], Ewing *et al* [5], and Peaceman [8]). The diffusion coefficient $k = k(S)$ is the conductivity of the medium, which is assumed here to depend only on the value of the saturation S. The fractional flow function $f = f(S)$ determines the transport term $\nabla \cdot (f(S)\mathbf{u})$ where $\mathbf{u}$ is the Darcy velocity of the fluid. We assume in this analysis that $\mathbf{u}$ is given and sufficiently smooth. In reality, it is

[1] This work was supported in part by NSF Grant EHR-910-18772 and the Partnership In Computational Sciences Consortium of DOE through Martin Marietta Contracts 19X-5K965C and 19X-5K966V.

1991 Mathematics Subject Classification. 35B45, 35B65, 65M60

Key words and phrases. Porous media, transport, degenerate advection–diffusion, regularization.

obtained by solving another, possibly nonlinear, pressure equation. The term $Q = Q(S)$ represents the source/sink terms, while q is the boundary flux.

Equation (1.1) has been studied extensively by many authors, especially in the one dimensional, purely parabolic case with the conductivity k taking the special form $k(\xi) = \xi^\nu$(see Rose [9] and its references). In a subsequent paper [10], Rose investigated the more general equation (1.1) in one space dimension, with time-varying mixed boundary conditions, and a single degeneracy for k at the origin: requiring $k(\xi)$ to be bounded above and below by constant multiples of $|\xi|^\nu$. In addition, both f and k were assumed to be at least C^2 functions of S, and in some cases, implicitly assumed either k was monotone increasing or concave. The proofs also required that f' should vanish at each degeneracy of k. In [10], it is stated that the results provided can be used to handle the case of two degeneracies (S=0, 1) which occur in two-phase immiscible flows of groundwater in certain soils. However, the proofs implicitly assume that $f(0) = f(1)$ (see (2.18)-(2.19) of that paper) which precludes the application of those results to standard fractional flows f which have the characteristic 'S'-shaped graph.

D. L. Smylie, in his doctoral thesis [12], studies the purely parabolic case in several dimensions and generalizes the results of [9,11] to the case of two degeneracies and relaxes the controlled decay to just lower bounds on k: for example, for ξ near 0, $k(\xi) \geq c\,|\xi|^\nu$. It is along the lines of [10,12] that we follow in our analysis, i.e. our primary goal is to numerically approximate the solution to the full multidimensional advection–diffusion equation (1.1) with two degeneracies by a discrete Galerkin method. This paper focuses on the necessary *a priori* and regularity estimates required for this approach. In a forthcoming paper we apply these estimates to obtain improved Galerkin approximations.

The multidimensional extensions complicate somewhat the derivation of the desired estmates, but in many cases indicate more direct routes. In one spatial dimension, a simple explicit operation was used to invert the second order diffusion term (integrate twice). In the multidimensional setting, this corresponds to integration against Green's functions which is typically used for "local" as opposed to Sobolev norm estimates and may also be rather technical. Using the approach in Section 2 (see also [10,12]), we are often able to provide more direct proofs of somewhat greater generality. In particular, we relax the hypotheses to include the familar 'S' shaped fractional flow curves f appearing in two phase applications which are formulated in terms of total velocity and pressure [8]. Our only constraints on k are that it be bounded and regulated from below:

$$k(s) \geq \begin{cases} c_1|s|^{\nu_1} & 0 \leq s \leq \alpha_1 \\ c_2 & \alpha_1 \leq s \leq \alpha_2 \\ c_3|1-s|^{\nu_2} & \alpha_2 \leq s \leq 1 \end{cases} \tag{1.2}$$

where the c_j $(1 \leq j \leq 3)$ are positive constants, and the parameters $\alpha_j (j = 1, 2)$, ν_2, and ν_1 satisfy $0 < \alpha_1 < \frac{1}{2} < \alpha_2 < 1$ and $0 < \nu_1, \nu_2 \leq 2$. We then define

$$\nu := \max(\nu_1, \nu_2). \tag{1.3}$$

For error estimates over bounded domains Ω, in effect the parameters ν_j may be replaced by ν. For purposes which will become clear in dealing with various Lebesgue estimates, we define γ as the Hölder conjugate index to ν,

$$\gamma := \frac{2+\nu}{1+\nu}\;. \tag{1.4}$$

For f, we require that there be a positive constant η such for all $0 \le a \le b \le 1$,

$$\eta \ |f(b) - f(a)|^2 \le \Big(K(b) - K(a)\Big)(b-a) \tag{1.5}$$

where K is the antiderivative of k,

$$K(S) := \int_0^S k(r)\, dr \ . \tag{1.6}$$

These conditions control the degeneracy of k and, in a sense, indicate diffusion-dominated transport.

This paper is structured as follows. In section 2, we fix the notation and, for completeness, give preliminary results useful in the remainder of the paper. In section 3, we derive *a priori* inequalities for the solution to (1.1). We shall assume there exists a weak solution S to (1.1) on $\Omega \times [0, T_0]$ as, for example, in Gilding and Peletier [7]:

$$-\int_0^{T_0} (S, \phi_t)(\tau)\, d\tau - \int_0^{T_0} (f(S)\mathbf{u}, \nabla\phi)(\tau)\, d\tau + \int_0^{T_0} (k(S)\nabla S, \nabla\phi)(\tau)\, d\tau$$

$$= (S^0, \phi(\cdot, 0)) + \int_0^{T_0} (Q(S), \phi)(\tau)\, d\tau - \int_0^{T_0}\int_{\partial\Omega} q\phi\, d\sigma\, d\tau$$

holds for all $\phi \in C^1([0, T_0], C^2(\Omega))$ such that $\phi(x, T_0) = 0$ on Ω. We then prove the well–posedness of the nonlinear problem (uniqueness and continuous dependence of the solution on the data). Section 4 contains the perturbation of (1.1) to obtain a nondegenerate equation

$$\frac{\partial S_\beta}{\partial t} + \nabla \cdot (f(S_\beta)\mathbf{u}) - \nabla \cdot (k_\beta(S_\beta)\nabla S_\beta) = Q$$

where k_β is conveniently defined. We show the solution S_β of this regularized problem converges in appropriate functional spaces to the solution of our original problem, establishing continuity with respect to coefficient data. Finally, we establish regularity results for the solution of the perturbed problem.

In the remainder of this section, we set our notation. For simplicity we assume the domain Ω has unit Lebesgue measure and denote by

$$S_\Omega := \int_\Omega S$$

the mean value of the function S over Ω. We denote by $(S_1, S_2)_\Omega := \int_\Omega S_1(x)S_2(x)dx$, when this has meaning. Often we will omit the subscript Ω on the integral if there is no ambiguity. We use the notation

$$\|S\|_{L^p} := \|S\|_{L^p(\Omega)} := (\int_\Omega |S|^p)^{\frac{1}{p}}$$

for the norm of the Lebesque spaces and say that $S \in L^p(\Omega)$ when this expression is finite. The mixed norm spaces consist of all S which are Lebesgue measurable on $\Omega \times [0, T_0]$ and for which the norm

$$\|S\|_{L^p(L^q)} := \|S\|_{L^p(0,T_0,L^q(\Omega))} := (\int_0^{T_0} \|S\|^p_{L^q}(\tau)\, d\tau)^{\frac{1}{p}}$$

is finite. The Sobolev spaces $W_r^p(\Omega)$ are defined according to

$$W_r^p(\Omega) := \{S \in L^p(\Omega) : D^\alpha S \in L^p(\Omega) \text{ for } |\alpha| \le r\}$$

where D^α is the α-th weak derivative, α is a standard mulit-index $(\alpha_1, \ldots, \alpha_n)$, and a norm is provided by

$$\|S\|_{W_r^p} := (\sum_{|\alpha|\le r} \|D^\alpha S\|_{L^p}^p)^{\frac{1}{p}}.$$

When $p = 2$, we write $H^r(\Omega)$ for $W_r^2(\Omega)$. The homogeneous space $W_r^p(\Omega)_0$ is the completion of $C_0^\infty(\Omega)$ in $W_r^p(\Omega)$ and the "negative" Sobolev space $H^{-1}(\Omega)$ is the Banach space dual of $H^1(\Omega)$. We will also have occasion to use mixed Lebesgue-Sobolev norms

$$\|S\|_{L^p(H^j)} := \|S\|_{L^p([0,T_0],H^j(\Omega))} := (\int_0^{T_0} \|S\|_{H^j}^p)^{\frac{1}{p}}.$$

The symbols C and c are used to denote positive constants which are independent of parameters such as β, x, and t, but which may change from line to line in the different estimates.

To close this section, the authors would like to take this opportunity to express their sincere thanks to Richard Ewing for posing this study and providing several key references.

2. Preliminaries: The Poisson Solution Operator T

We give in this section elementary results required in later sections. Several *a priori* and error estimates in Section 3 are formulated in terms of the negative Sobolev space H^{-1}. We use the solution operator T defined below in order to provide a more convenient equivalent norm for $(H^1)^*$ and specify its relationship to the Lebesgue spaces. Required properties of this operator which may be more or less well known are also recounted briefly for completeness. Within this section, we use the notation f (and g) to denote a distribution in $(H^1)^*$ which should not be confused with the coefficient f appearing in the equation (1.1). The notation ϕ (and ω) is used for members of H^1.

Let $f \in (H^1)^*$, then $f_\Omega := (f, 1)$ in the sense of the $((H^1)^*, H^1)$ duality, which coincides with $\int_\Omega f dx$ when f is Lebesgue integrable on Ω. Consider the elliptic Neumann boundary value problem:

$$\begin{aligned} -\Delta\omega &= f - f_\Omega && \text{in } \Omega \\ \frac{\partial\omega}{\partial n} &= 0 && \text{on } \partial\Omega \\ \omega_\Omega &= f_\Omega, && \end{aligned} \tag{2.1}$$

which (see Ciarlet [3]) has a unique solution $\omega \in H^1$. We define the solution operator $T : (H^1)^* \to H^1$ by setting for each $f \in (H^1)^*$, $T(f) = \omega$, where $\omega \in H^1$ is the unique solution to (2.1). A weak formulation of (2.1) is given by

$$\begin{aligned} (\nabla\omega, \nabla\phi) &= (f, \phi) - (f_\Omega, \phi) \\ &= (f, \phi) - f_\Omega\, \phi_\Omega \\ &= (f, \phi) - (Tf)_\Omega\, \phi_\Omega \end{aligned} \tag{2.2}$$

for all $\phi \in H^1$. Thus, the duality relationship may be written

$$(f, \phi) = (\nabla(Tf), \nabla\phi) + f_\Omega \, \phi_\Omega. \tag{2.3}$$

In particular, if we take $\phi = Tg$ with $g \in H^{-1}$, we may rewrite this to obtain

$$(f, Tg) = (\nabla Tf, \nabla Tg) + f_\Omega \, (Tg)_\Omega = (\nabla Tf, \nabla Tg) + f_\Omega \, g_\Omega \tag{2.4}$$

which we show may be considered as the inner product on H^{-1}. If we set $g = f$, then

$$(f, Tf) = \|\nabla Tf\|^2_{L^2} + (f_\Omega)^2 = \|\nabla Tf\|^2_{L^2} + (Tf)^2_\Omega. \tag{2.5}$$

Proposition 2.1 The solution operator T is a linear, symmetric, positive definite operator. Moreover, the equivalent expressions in the identity (2.4) define an inner product on H^{-1} with norm given by the square root of any one of the equivalent expressions in (2.5).

Proof. The argument that T is linear is standard since Tf is the unique solution to the linear equation (2.1). The fact that T is symmetric follows from the symmetry in identlity (2.4) of f and g. It is obvious from (2.5) that T is a nonnegative operator. If $(f, Tf) = 0$, then equation (2.5) shows that both ∇Tf and $(Tf)_\Omega$ vanish which forces $Tf \equiv 0$. But the definition of T shows that $f - f_\Omega = -\Delta(Tf) = 0$ and so f must be the constant f_Ω. But the condition $(Tf)_\Omega = f_\Omega$ on T implies that $f \equiv 0$ and so T is positive definite. □

Observe that H^1 may be normed with

$$|||\phi|||_{H^1} := \left(\|\nabla\phi\|^2_{L^2} + (\phi_\Omega)^2 \right)^{\frac{1}{2}} \tag{2.6}$$

which is equivalent (up to constant multiples) with the usual norm on H^1 given by:

$$\|\phi\|_{H^1} := \left(\|\nabla\phi\|^2_{L^2} + \|\phi\|^2_{L^2} \right)^{\frac{1}{2}}. \tag{2.7}$$

That the quantity in (2.6) is no larger than that in (2.7) follows directly from Hölder's inequality applied to ϕ_Ω and the fact that Ω has unit measure. The opposite inequality follows from a version of Poincaré's inequality: $\|\phi\|_{L^2} \le c\, |||\phi|||_{H^1}$.

From the definition of the norm $|||\cdot|||$ and the identity (2.5), we have that

$$(Tf, f) = |||Tf|||^2_{H^1}. \tag{2.8}$$

Proposition 2.1 indicates that $(Tf, f)^{\frac{1}{2}}$ is a natural norm on $(H^1)^*$. It is in fact an easy exercise (use Hölder's inequality for ℓ^p_{n+1}) to show that this is the Banach dual norm for $(H^1, |||\cdot|||)$. Along a different line, the Sobolev embedding lemma (see [1]) implies that $H^1(\Omega)$ is continuously embedded in $L^p(\Omega)$, $p \le 6$ since $\Omega \subset \mathbb{R}^n$, $n \le 3$, and Ω has measure one. The duality argument

$$\begin{aligned} \|f\|^2_{(H^1)^*} &= (Tf, f) \le \|Tf\|_{L^6} \|f\|_{L^{6/5}} \\ &\le c\, \|Tf\|_{H^1} \|f\|_{L^{6/5}} = c\, \|f\|_{(H^1)^*} \|f\|_{L^{6/5}} , \end{aligned}$$

shows that $L^{6/5}$ is continuously embedded in H^{-1}. We summarize these statements in the following Proposition.

Proposition 2.2 The Sobolev space $H^1(\Omega)$ may be equivalently renormed by (2.6), in which case, the dual norm is given by

$$\begin{aligned} \|f\|_{(H^1)^*} &= (Tf,f)^{\frac{1}{2}} \\ &= |||Tf|||_{H^1} \,, \end{aligned} \tag{2.9}$$

and $(H^1)^*$ is naturally identified with H^{-1}. Moreover, for $n \leq 3$ the Lebesgue space $L^{6/5}(\Omega)$ is continuously embedded in $H^{-1}(\Omega)$.

In the sections that follow, we will need the fact that the inverse operator T commutes with time derivatives. Although not required in this paper, under a no-flow boundary condition, the operator also commutes with the Laplacian. The proof follows by the facts that the commutator of T and $\frac{\partial}{\partial t}$ applied to any (smooth) ϕ satisfies (2.1) with homogeneous data and that the solution of the boundary value problem is unique (cf. Fadimba [6]).

Proposition 2.3 For smooth ϕ, T and $\frac{\partial}{\partial t}$ commute, i.e

$$\frac{\partial}{\partial t}(T\phi) = T(\frac{\partial \phi}{\partial t}) \tag{2.10}$$

Moreover, if $\phi \in H^1(\Omega)$, and

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{2.11}$$

then

$$T(\Delta\phi) = \Delta(T\phi). \tag{2.12}$$

3. A Priori Estimates

In this section we establish *a priori* inequalities for the solutions and show that problem (1.1) is well–posed. For simplicity, we begin by assuming that $Q(S) \equiv 0$ and $q \equiv 0$ which leads to continuity estimates with respect to initial data. We discuss in Theorem 3.5 how the proof is modified to handle the general case.

In addition to the multivariate generalization presented here, we are able to relax the restrictions on k and f and do not require a direct evaluation of $T(\nabla\cdot(f(S)\mathbf{u})-\nabla\cdot k(S)\nabla S)$ or similar terms as in [9,10],and are able to exploit the symmetry of the special elliptic inversion operator T to simplify estimates.

In this paper, property (1.5) plays a crucial role, enabling us to control the transport term in our estimates. We consider first the homogeneous problem

$$\frac{\partial S}{\partial t} + \nabla\cdot(f(S)\mathbf{u}) - \nabla\cdot(k(S)\nabla S) = 0 \qquad \text{on} \quad \Omega\times[0,T_0], \tag{3.1.a}$$

$$f(S)\,\mathbf{u}\cdot\mathbf{n} - \frac{\partial}{\partial \mathbf{n}}K(S) = 0 \qquad \text{on} \quad \partial\Omega\times[0,T_0], \tag{3.1.b}$$

$$S(x,0) = S^0(x) \geq 0 \qquad \text{on} \quad \Omega. \tag{3.1c}$$

Here we assume as before that $k(s) \geq 0$ and $0 \leq S^0(x) \leq 1$. We denote by $\partial\Omega$ the boundary of Ω and by $\mathbf{n}$ its outward unit normal.

The following elementary lemma will be used to assist establishing Lebesgue norm estimates from our primary inequalities (compare pp. 7-10 of [12]).

Lemma 3.1 If k satisfies the condition (1.2) and $\nu := \max(\nu_2, \nu_1)$, then there is a positive constant C^* depending only on the parameters for k so that for all $0 \leq a \leq b \leq 1$ there holds

$$C^* \, |b-a|^{1+\nu} \leq \, K(b) - K(a). \tag{3.2}$$

Proof. Observe that inequality (1.2) is equivalent to

$$k(s) \geq \begin{cases} \tilde{c}_1 |s|^{\nu_1} & 0 \leq s \leq \alpha_2 \\ \tilde{c}_3 |1-s|^{\nu_2} & \alpha_1 \leq s \leq 1 \, . \end{cases} \tag{3.3}$$

Consider first the case $0 \leq a < b \leq \alpha_2$. By Hölder's inequality it follows that $\left(s_1^{\nu_1+1} + s_2^{\nu_1+1}\right) \leq (s_1 + s_2)^{\nu_1+1}$, and so, with $s_2 = a$, $s_1 = b - a$ and inequality (3.3) we have

$$\begin{aligned} (b-a)^{1+\nu_1} \;&\leq\; b^{1+\nu_1} - a^{1+\nu_1} \;=\; (1+\nu_1) \textstyle\int_a^b s^{\nu_1} ds \\ &\leq\; 3\tilde{c}_1 \textstyle\int_a^b k(s) ds \;=\; 3\tilde{c}_1 (K(b) - K(a)) \, . \end{aligned} \tag{3.4}$$

Inequality (3.2) then follows since $\nu \geq \nu_1$. The case $\alpha_1 \leq a \leq b \leq 1$ follows similarly.

For the remaining case $0 \leq a < \alpha_1 < \alpha_2 < b \leq 1$, we recall that $k(s) > 0$ on $(0,1)$ and so from (1.2)

$$K(b) - K(a) \geq K(\alpha_2) - K(\alpha_1) \geq c_2 \, (\alpha_2 - \alpha_1).$$

Hence if we define $\alpha_0 := c_2 \, (\alpha_2 - \alpha_1)$, then

$$(b-a)^{1+\nu} \leq b - a \leq \frac{1}{\alpha_0} \Big(K(b) - K(a) \Big) \tag{3.5}$$

since $0 \leq b - a \leq 1$. $\square$

Before proceeding with our estimates, we provide in the following proposition relationships between conditions on the coefficients. Inequalities such as (1.5) appear within the bodies of proofs in earlier works (compare Theorem 2.1 of [10] with $k(S) \sim S^\nu$ and implicit assumptions on f for valid applications of L'Hospital's rule).

Proposition 3.2 Suppose k satisfies condition (1.2).

i.) If $f \in C^1[0,1]$ and $f'(0) = f'(1) = 0$ with f' Lipschitz at 0 and 1, then there is a positive constant η so that property (1.5) relating f and k holds for all $0 \leq a \leq b \leq 1$, that is

$$\eta \, |f(b) - f(a)|^2 \;\leq\; \Big(K(b) - K(a) \Big)(b-a) \, .$$

ii.) Conversely, if f and k satisfy condition (1.5), then $|f'|^2 \le ck$, and so f' vanishes with k.

iii.) If $f(0) = f(1) = 0$, then there is a positive constant C such that

$$|f(s)| \le C\sqrt{k(s)} \qquad \text{for all } 0 \le s \le 1.$$

Proof. To prove part i.), consider first the case $0 \le a < \alpha_1 < \alpha_2 < b \le 1$. From (3.5) we have

$$\alpha_0 \le \frac{K(b) - K(a)}{b - a}$$

and so

$$\frac{(f(b) - f(a))^2}{(b-a)(K(b) - K(a))} = \Big(\frac{f(b) - f(a)}{b-a}\Big)^2 \frac{1}{\frac{K(b)-K(a)}{b-a}} \le \frac{\|f'\|_\infty^2}{\alpha_0}$$

which yields inequality (1.5) if $\eta \ge \alpha_0/\|f'\|_\infty^2$.

Next consider the case $0 \le a < b \le \alpha_2$. By inequality (3.4) together with an application of the mean value theorem there exist c, d between a and b such that

$$\begin{aligned}\frac{|f(b) - f(a)|^2}{(b-a)(K(b) - K(a))} &\le \frac{1}{\tilde{c}_1} \frac{|f(b) - f(a)|^2}{(b-a)(b^{\nu_1+1} - a^{\nu_1+1})} \\ &\le \frac{1}{\tilde{c}_1} \frac{f'(c)^2}{d^{\nu_1}}.\end{aligned}$$

But s^{ν_1} increases, so $d > \frac{c}{2}$ and hence

$$\begin{aligned}\frac{f'(c)^2}{d^{\nu_1}} &\le 2^{\nu_1} \frac{f'(c)^2}{c^{\nu_1}} \\ &\le 4c^{2-\nu_1} \Big[\frac{f'(c)}{c}\Big]^2 \\ &\le 4C\end{aligned}$$

where the last inequality follows from the properties assumed of f (i.e., $f'(0) = 0$ and f' is Lipschitz at 0) and the fact that $0 \le \nu_1 \le 2$. By combining these last two inequalities, the Proposition follows for this range of a, b if η is set greater or equal to the reciprocal of $\frac{4C}{\tilde{c}_1}$.

For the case $\alpha_1 \le a < b \le 1$, we only need change variables ($\xi = 1 - s$) and replace ν_1 by ν_2 in the argument above.

Part ii.) of the Proposition follows by letting $b = s + h$, $a = s$ and letting h tend to zero, while part iii.) is verified by first observing that the condition (1.2) implies that k satisfies inequality (3.3). So, if $0 \le s \le \alpha_2$, we have

$$\begin{aligned}\frac{f(s)^2}{k(s)} &\le \frac{f(s)^2}{\tilde{c}_1 s^{\nu_1}} \le c\, s^{2-\nu_1} \Big(\frac{f(s)}{s}\Big)^2 \\ &\le c\|f'\|_{L^\infty}^2 =: C.\end{aligned}$$

where we have used the fact that $f(0) = 0$ and applied the mean value theorem. A similar proof holds for the case $\alpha_1 \le s \le 1$, thereby establishing the proposition. □

Lemma 3.3 Assume that k and f satisfy conditions (1.2) and (1.5) and that S_1 and S_2 solve the homogeneous boundary value problem (3.1) with initial values S_1^0 and S_2^0, respectively, which satisfy $\int_\Omega S_1^0 = \int_\Omega S_2^0$, then

$$\sup_{0\le t\le T_0} \|S_1 - S_2\|_{H^{-1}}^2 + \iint_{\Omega\times[0,T_0]} \Big(K(S_1) - K(S_2)\Big)(S_1 - S_2)\, dxdt \le C_1 \|S_1^0 - S_2^0\|_{H^{-1}}^2 . \quad (3.6)$$

Proof. First we note for each t the general fact that the average saturation is constant in time under the conditions that the source and boundary terms vanish, i.e., $S_j(\cdot,t)$ has the same mean value over Ω as S_j^0. To see this we integrate (3.1a) over Ω and get

$$\frac{d}{dt}\int_\Omega S + \int_\Omega \nabla\cdot(f(S)\mathbf{u} - \nabla K(S))dx = 0 .$$

An application of the divergence theorem gives

$$\frac{d}{dt}\int_\Omega S + \int_{\partial\Omega} (f(S)\mathbf{u}\cdot\mathbf{n} - \frac{\partial}{\partial\mathbf{n}}K(S)) = 0 ,$$

which combined with the boundary condition (3.1b) implies $\frac{d}{dt}\int_\Omega S = 0$. Hence as stated

$$\int_\Omega S = \int_\Omega S^0.$$

We apply the partial differential equation (3.1a) to both S_1,S_2 and subtract to obtain

$$\frac{\partial}{\partial t}(S_1 - S_2) + \nabla\cdot(f(S_1) - f(S_2))\mathbf{u} \;-\; \Delta(K(S_1) - K(S_2)) = 0.$$

Applying the operator T to this equation and using the fact (2.10) that T and $\frac{\partial}{\partial t}$ commute, we have

$$\frac{\partial}{\partial t}T(S_1 - S_2) + T\Big(\nabla\cdot(f(S_1) - f(S_2))\mathbf{u} - \Delta(K(S_1) - K(S_2))\Big) = 0. \quad (3.7)$$

But by recalling that $(T\phi,\phi) =: \|\phi\|_{H^{-1}}^2$, T is symmetric (see Proposition 2.1), and again that T and $\frac{\partial}{\partial t}$ commute, we observe that

$$\begin{aligned}\frac{d}{dt}\|\phi\|_{H^{-1}}^2 &= \Big(\frac{\partial}{\partial t}T\phi,\phi\Big) + \Big(T\phi,\frac{\partial}{\partial t}\phi\Big)\\ &= 2\Big(\frac{\partial}{\partial t}T\phi,\phi\Big).\end{aligned} \quad (3.8)$$

Multiplying equation (3.7) by $S_1 - S_2$ and integrating over Ω, we may substitute $S_1 - S_2$ for ϕ in (3.8) in order to rewrite the first term of the resulting equation from (3.7) as a derivative of a norm. For the second term of the equation, we use the symmetry of T to pass it over to the term $S_1 - S_2$ and follow by applying integration by parts to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|S_1 - S_2\|_{H^{-1}}^2 + &\Big(K(S_1) - K(S_2), S_1 - S_2\Big)\\ &= \Big([f(S_1) - f(S_2)]\,\mathbf{u}, \nabla T(S_1 - S_2)\Big).\end{aligned} \quad (3.9)$$

Here we have used the Neumann-boundary condition (3.1b) to eliminate the boundary term

$$\int_{\partial\Omega} \Big[(f(S_1)-f(S_2))\mathbf{u} - \nabla(K(S_1)-K(S_2))\Big]\cdot \mathbf{n}\ T(S_1-S_2)\,d\sigma$$

while (2.1) is used to observe that the term $\frac{\partial T(S_1-S_2)}{\partial \mathbf{n}}$ vanishes on $\partial\Omega$. The final term, as yet unaccounted for, is $\Big(K(S_1)-K(S_2),(S_1-S_2)_\Omega\Big)_\Omega$ which arose from the properties of the solution operator T and the term $\Big(K(S_1)-K(S_2),\Delta T(S_1-S_2)\Big)_\Omega$. But we recall from above that by our assumption it follows that $(S_2-S_1)_\Omega=(S_2^0-S_1^0)_\Omega=0$ and so this last term vanishes as well.

We now concentrate on estimating the right handside of (3.9). We may apply Hölder's inequality followed by the arithmetic–geometric mean inequality to obtain

$$\begin{aligned}
\Big|\Big([f(S_1)-f(S_2)]\mathbf{u},\nabla T(S_1-S_2)\Big)\Big| &\le \|\mathbf{u}\|_\infty\,\|f(S_1)-f(S_2)\|_{L^2}\,\|\nabla T(S_1-S_2)\|_{L^2}\\
&\le \|\mathbf{u}\|_\infty\,\|f(S_1)-f(S_2)\|_{L^2}\,|T(S_1-S_2)|_{H^1}\\
&\le \frac{\eta}{2}\|f(S_1)-f(S_2)\|_{L^2}^2+\frac{\|\mathbf{u}\|_\infty^2}{2\,\eta}|T(S_1-S_2)|_{H^1}^2\ .
\end{aligned} \tag{3.10}$$

where η is the constant in (1.5). Using property (1.5) with $b=S_1$ and $a=S_2$, we get for each time t

$$\begin{aligned}
&\Big|\Big([f(S_1)-f(S_2)]\mathbf{u},\nabla T(S_1-S_2)\Big)\Big|\\
&\qquad\le\ \frac{1}{2}\Big(K(S_1)-K(S_2),S_1-S_2\Big)+\frac{\|\mathbf{u}\|_\infty^2}{2\,\eta}|T(S_1-S_2)|_{H^1}^2\ .
\end{aligned} \tag{3.11}$$

Hence, combining this inequality with the identity (3.9) and taking the inner product from the right hand side to the like term on the left, we have

$$\begin{aligned}
\frac{d}{dt}\|S_1-S_2\|_{H^{-1}}^2+\Big(K(S_1)-K(S_2),S_1-S_2\Big) &\le C\,\|\mathbf{u}\|_\infty^2\,|T(S_1-S_2)|_{H^1}^2\\
&\le C\,\|\mathbf{u}\|_\infty^2\,\|S_1-S_2\|_{H^{-1}}^2
\end{aligned} \tag{3.12}$$

where the last inequality follows from the Proposition 2.2. Inequality (3.6) now follows by applying Grönwall's Lemma. □

The following Theorem shows that the requirement that the initial conditions S_1^0, S_2^0 have the same mean over Ω can be eliminated. Additional norm estimates also follow immediately which improve upon those in [12].

Theorem 3.4 Assume that k and f satisfy conditions (1.2) and (1.5) and that S_1 and S_2 each solve the homogeneous boundary value problem (3.1) with initial values S_1^0 and S_2^0, respectively, then

$$\sup_{0\le t\le T_0}\|S_1-S_2\|_{H^{-1}}^2+\iint_{\Omega\times[0,T_0]}\Big(K(S_1)-K(S_2)\Big)(S_1-S_2)\,dxdt\ \le\ C_1\,\|S_1^0-S_2^0\|_{H^{-1}}^2, \tag{3.13}$$

$$\|S_1 - S_2\|_{L^\infty(H^{-1})} \le C_1 \, \|S_1^0 - S_2^0\|_{H^{-1}}, \tag{3.14}$$

$$\|K(S_1) - K(S_2)\|_{L^2(L^2)} \le C_1 \, \|k\|_\infty \, \|S_1^0 - S_2^0\|_{H^{-1}} \ , \tag{3.15}$$

$$\|S_2 - S_1\|^{2+\nu}_{L^{2+\nu}(L^{2+\nu})} \le C_1 \, \|S_2^0 - S_1^0\|^2_{H^{-1}} \tag{3.16}$$

where $\nu = \max(\nu_1, \nu_2)$. In particular, if γ is given by (1.4), then

$$\|S_2 - S_1\|^{2+\nu}_{L^{2+\nu}(L^{2+\nu})} \le C_1 \, \|S_2^0 - S_1^0\|^2_{L^\gamma} \tag{3.17}$$

Proof. The only real change required of the proof of (3.6) given in Lemma 3.3 is that an additional term must be included in equation (3.9):

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|S_1 - S_2\|^2_{H^{-1}} + \Big(K(S_1) - K(S_2), S_1 - S_2\Big) \\ = \Big((f(S_1) - f(S_2))\mathbf{u}, \nabla T(S_1 - S_2)\Big) \quad + \quad (K(S_1) - K(S_2))_\Omega (S_1 - S_2)_\Omega. \end{aligned} \tag{3.18}$$

and must be estimated. We may again use the arithmetic-geometric mean inequality to estimate the additional term:

$$(K(S_1) - K(S_2))_\Omega (S_1 - S_2)_\Omega \le \frac{\epsilon_1}{2}(K(S_1) - K(S_2))_\Omega^{\,2} + \frac{1}{2\epsilon_1}(S_1 - S_2)_\Omega^{\,2}. \tag{3.19}$$

The first term on the right hand side of (3.19) may be bounded using

$$\begin{aligned} (K(S_1) - K(S_2))_\Omega^{\,2} &\le \|K(S_1) - K(S_2)\|^2_{L^2} \\ &\le \|k\|_\infty (K(S_1) - K(S_2), S_1 - S_2) \end{aligned} \tag{3.20}$$

by the Cauchy-Schwarz inequality and by the facts that K increases and for each $a, b \ge 0$

$$(K(b) - K(a))^2 \le \|k\|_\infty (K(b) - K(a)) \, (b - a). \tag{3.21}$$

By choosing $\epsilon_1 = (2\|k\|_\infty)^{-1}$ and using the estimate (3.20) and Proposition 2.1, inequality (3.19) becomes

$$\Big(K(S_1) - K(S_2)\Big)_\Omega (S_1 - S_2)_\Omega \le \frac{1}{4}\Big(K(S_1) - K(S_2), S_1 - S_2\Big) + \|k\|_\infty \, \|S_1 - S_2\|^2_{H^{-1}} \tag{3.22}$$

Inequality (3.11) remains as before, hence the right hand side of (3.18) may be estimated by

$$\frac{3}{4}\Big(K(S_1) - K(S_2), S_1 - S_2\Big) + \left(\|k\|_\infty + \frac{C\|\mathbf{u}\|^2_\infty}{2}\right) \|S_1 - S_2\|^2_{H^{-1}}.$$

The first term of this last expression may be 'buried' in the like term on the left hand side of (3.18) as before to give

$$\begin{aligned} \tfrac{d}{dt}\|S_1 - S_2\|^2_{H^{-1}} \quad + \quad \tfrac{1}{2}\Big(K(S_1) - K(S_2), S_1 - S_2\Big) \le \\ (C\,\|\mathbf{u}\|^2_\infty + 2\,\|k\|_\infty)\,\|S_1 - S_2\|^2_{H^{-1}}. \end{aligned} \tag{3.23}$$

Inequality (3.13) follows by applying Grönwall's Lemma.

Inequality (3.14) derives directly from (3.13), while inequality (3.15) uses both inequalties (3.13) and (3.21). Inequality (3.16) requires the integration over $\Omega \times [0, T_0]$ of the inequality

$$C^* |S_2 - S_1|^{2+\nu} \leq \Big(K(S_2) - K(S_1)\Big)(S_2 - S_1)$$

obtained from Lemma 3.1, followed by an application of inequality (3.13). The final inequality (3.17) follows immediately from (3.16) and by, according to Proposition 2.2, the continuous embeddings $L^\gamma \subset L^{6/5} \subset H^{-1}$ since $6/5 \leq \gamma = \frac{2+\nu}{1+\nu}$ when $0 \leq \nu \leq 2$. $\square$

We note that we have the usual Grönwall dependence of the constants in Lemma 3.3 and Theorem 3.4

$$C_1 = \exp\left(2T_0\left(\frac{c}{\eta}\,\|\mathbf{u}\|_\infty^2 + 2\|k\|_\infty\right)\right)$$

where c depends upon the parameters for k in condition (1.2).

Having established the preliminary lemmas and estimates for *a-priori* inequalities in the homogeneous case, we now establish the general result.

Theorem 3.5 Suppose that the coefficients k and f satisfy the assumptions (1.2) and (1.5). If, for $j = 1, 2$, S_j solves the boundary value problem (1.1) with initial value S_j^0, source/sink term Q_j, and boundary term q_j, then

$$\begin{aligned}\sup_{0\leq t\leq T_0} \|S_2 - S_1\|_{H^{-1}}^2 &+ \int_0^{T_0} \Big(K(S_2) - K(S_1), S_2 - S_1\Big)(\tau)\, d\tau \\ &\leq C\left\{\|S_2^0 - S_1^0\|_{H^{-1}}^2 + \int_0^{T_0}\Big(\|q_2 - q_1\|_{L^2(\partial\Omega)}^2 + \|Q_2 - Q_1\|_{H^{-1}}^2\Big)(\tau)\, d\tau\right\}.\end{aligned} \tag{3.24}$$

Proof. Following the proofs in Lemma 3.3 and Theorem 3.4, a version of inequality (3.18) remains valid if two additional terms (arising from the source/sink term and divergence theorem boundary terms)

$$-\int_{\partial\Omega}(q_2 - q_1)T(S_2 - S_1)d\sigma + \Big(Q_2 - Q_1, T(S_2 - S_1)\Big)_\Omega \tag{3.25}$$

are added to the right hand side. Using Cauchy's inequality, followed by the trace theorem for Sobolev spaces (Adams [1]), Proposition 2.2, and the arithmetic-geometric mean inequality, we can bound the first term as follows

$$\begin{aligned}\textstyle\int_{\partial\Omega}(q_2 - q_1)T(S_2 - S_1)d\sigma &\leq \|q_2 - q_1\|_{L^2(\partial\Omega)}\|T(S_2 - S_1)\|_{L^2(\partial\Omega)} \\ &\leq C\,\|q_2 - q_1\|_{L^2(\partial\Omega)}\|T(S_2 - S_1)\|_{H^1(\Omega)} \\ &\leq C\,\|q_2 - q_1\|_{L^2(\partial\Omega)}\|(S_2 - S_1)\|_{H^{-1}(\Omega)} \\ &\leq C\Big(\|q_2 - q_1\|_{L^2(\partial\Omega)}^2 + \|S_2 - S_1\|_{H^{-1}}^2\Big).\end{aligned}$$

The second term of (3.25) is bounded through duality by

$$\Big|\Big(Q_2 - Q_1, T(S_2 - S_1)\Big)\Big| \leq \|Q_2 - Q_1\|_{H^{-1}}\,|||T(S_2 - S_1)|||_{H^1}$$

which by Proposition 2.2 and the arithmetic-geometric mean inequality is in turn bounded by

$$C(\|Q_2 - Q_1\|^2_{H^{-1}} + \|S_2 - S_1\|^2_{H^{-1}}) \ .$$

Adding these two bounds to the right hand side of (3.23) and applying the Grönwall Lemma yields the desired result. □

Remark 3.6 We note that the simple arguments in the last paragraph of the proof of Theorem 3.4 show that the inequalities (3.14)–(3.16) hold in the nonhomogeneous case as well if the expression $\|S_2^0 - S_1^0\|_{H^{-1}}$ is replaced by

$$\left\{ \|S_2^0 - S_1^0\|^2_{H^{-1}} + \int_0^{T_0} \Big(\|q_2 - q_1\|^2_{L^2(\partial\Omega)} + \|Q_2 - Q_1\|^2_{H^{-1}} \Big)(\tau)\, d\tau \right\}^{\frac{1}{2}} \ .$$

In the case that $Q_j(S) = \lambda\, S_j$ and $q_1 = q_2$, these techniques can be used to show that the additional terms can be eliminated from the right hand side of the inequalities, i.e. that (3.14)–(3.16) hold in their original form with possibly different constants.

The following result will be particularly helpful in the next section which involves regularizing the equation in preparation for numerical treatments. The previous results of this section involve regularity with respect to data, including initial conditions, and source terms. This next result involves weighted regularity in terms of the coefficient k.

Theorem 3.7 Suppose k satisfies condition (1.2) and f is bounded. If the solution S to equation (1.1), with both $Q, q \equiv 0$, is sufficiently smooth, then

$$\|S\|^2_{L^\infty(L^2)} + \left\| \sqrt{k(S)}\ \nabla S \right\|^2_{L^2(L^2)} \le C \cdot T_0 + \|S^0\|^2_{L^2}. \tag{3.26}$$

Proof. We multiply equation (1.1a) by S and integrate over Ω to get:

$$\frac{1}{2}\frac{d}{dt}\|S\|^2_{L^2} + \Big(\nabla \cdot (\, f(S)\mathbf{u} - \nabla K(S)\,)\ ,\ S \Big)_\Omega = 0,$$

which, after applying the divergence theorem with boundary condition (1.1b), gives

$$\frac{1}{2}\frac{d}{dt}\|S\|^2_{L^2(\Omega)} + \Big(\nabla K(S), \nabla S \Big)_\Omega = \Big(f(S)\mathbf{u}, \nabla S \Big),$$

or, equivalently,

$$\frac{1}{2}\frac{d}{dt}\|S\|^2_{L^2(\Omega)} + \left\| \sqrt{k(S)}\ \nabla S \right\|^2_{L^2(\Omega)} = \Big(f(S)\mathbf{u}, \nabla S \Big). \tag{3.27}$$

If we set $F(S) := \int_0^S f(r)\, dr$, then we can rewrite and bound the right handside of (3.27) as follows:

$$\begin{aligned} \Big(f(S)\mathbf{u}\, , \nabla S \Big) &= \int_\Omega \mathbf{u} \cdot \nabla F(S(x,t))\, dx \\ &= \int_{\partial\Omega} F(S(x,t))\ \mathbf{u} \cdot \mathbf{n}\ d\sigma - \int_\Omega F(S(x,t))\ \nabla \cdot \mathbf{u}\ dx \\ &\le \|f(S)\|_{L^\infty} \Big(\|\mathbf{u}\|_{L^\infty}\ \mathrm{vol}_{n-1}(\partial\Omega) + |\mathbf{u}|_{W_1^1} \Big) \end{aligned}$$

where we have used integration by parts in the second equality. Combining this inequality with the identity (3.27), we get

$$\frac{d}{dt}\|S\|_{L^2}^2 + \left\|\sqrt{k(S)}\nabla S\right\|_{L^2}^2 \le C,$$

where $C = C(\mathbf{u})$, yielding inequality (3.26) upon integration over $[0, T_0]$. □

4. A Regularized Equation

Solutions to the degenerate problem (1.1) are not guaranteed to be sufficiently regular for numerical approximation purposes Rose [9], Smylie [12]. One approach is to regularize the equation to obtain a nondegenerate problem with corresponding smooth solutions. In this section we regularize the problem by perturbing the diffusion coefficient k in a convenient way. Here again we use to our advantage the symmetry of the solution operator T in order to obtain our norm estimates for the error $S_\beta - S$ and avoid dealing with explicit expressions of T, as occurred in the one dimensional case [9].

In particular, we replace k by $k_\beta > 0$ with $k_\beta \to k$ in some sense as $\beta \to 0$ and determine in what manner S_β converges to S, the solution to the original equation (1.1). For example, in Theorem 4.1 below, we specify specific rates of convergence in terms of the uniform norm of the error in K. These estimates provide continuity with respect to diffusion coefficient data and establish the existence of a solution S belonging to $L^{2+\nu}(L^{2+\nu})$ by considering a nonincreasing sequence of nondegenerate coefficients $\{k_{\beta_m}\}$ converging to k and applying inequality (4.7) below (with $k_\beta := k_{\beta_m}$ and $k := k_{\beta_n}$, $m \le n$) to the corresponding solutions to obtain a Cauchy sequence. For given k_β, we will set

$$K_\beta(S) := \int_0^S k_\beta(s)\, ds \ . \tag{4.1}$$

We also recall

$$\nu = \max(\nu_1, \nu_2), \tag{4.2}$$

and that

$$\gamma = \frac{2+\nu}{1+\nu}, \tag{4.3}$$

is the conjugate Hölder exponent to $2 + \nu$, that is

$$\frac{1}{\gamma} + \frac{1}{2+\nu} = 1.$$

We let S_β be the solution of the nondegenerate parabolic problem (1.1) with nondegenerate (and smooth, if desired) diffusion coefficient k_β replacing k:

$$\frac{\partial S_\beta}{\partial t} + \nabla \cdot f(S_\beta)\mathbf{u} - \nabla \cdot k_\beta(S_\beta)\nabla S_\beta = Q \quad \text{on } \Omega \times (0, T], \tag{4.4a}$$

$$f(S_\beta)\mathbf{u}\cdot\mathbf{n} - \frac{\partial}{\partial \mathbf{n}} K_\beta(S_\beta) = q \quad \text{on } \partial\Omega \times [0, T]\ , \tag{4.4b}$$

and

$$S_\beta(x, 0) = S^0(x) \ge 0 \quad \text{on } \Omega\ . \tag{4.4c}$$

Then S_β is known to be C^2 in the space variable x, and C^1 in t (cf. [9]).

Theorem 4.1 Assume that $k_\beta \geq k$ and that the coefficient k satisfies condition (1.2) and k, f satisfies condition (1.5). Let S, S_β be the solutions to original equation (1.1) and the regularized equation (4.4), respectively, with the same initial condition S^0. If $C(\beta) := c\,\|K - K_\beta\|_\infty^\gamma$ with c determined below, then

$$\sup_{0\leq t\leq T} \|S_\beta - S\|_{H^{-1}}^2 + \int_0^T \Big(K_\beta(S_\beta) - K_\beta(S), S_\beta - S \Big)_\Omega (\tau)\, d\tau \leq C(\beta), \tag{4.5}$$

$$\|K_\beta(S_\beta) - K_\beta(S)\|_{L^2(L^2)} \leq C(\beta)^{1/2}, \tag{4.6}$$

and

$$\|S_\beta - S\|_{L^{2+\nu}(L^{2+\nu})} \leq C(\beta)^{\frac{1}{2+\nu}}. \tag{4.7}$$

Proof. We subtract (1.1a) from (4.4a) to get

$$\frac{\partial}{\partial t}(S_\beta - S) + \nabla\cdot(f(S_\beta) - f(S))\mathbf{u} - \Delta(K_\beta(S_\beta) - K(S)) = 0.$$

Proceeding as in Section 3 (equations (3.7) through (3.9)) and noticing that $(S_\beta - S)_\Omega$ vanishes, we obtain

$$\frac{1}{2}\frac{d}{dt}\|S_\beta - S\|_{H^{-1}}^2 + \Big(K_\beta(S_\beta) - K(S), S_\beta - S \Big)_\Omega = \Big((f(S_\beta) - f(S))\mathbf{u}, \nabla T(S_\beta - S) \Big)_\Omega,$$

which can be rewritten as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|S_\beta - S\|_{H^{-1}}^2 &+ (K_\beta(S_\beta) - K_\beta(S), S_\beta - S)\\ &= -(K_\beta(S) - K(S), S_\beta - S) + ((f(S_\beta) - f(S))\mathbf{u}, \nabla T(S_\beta - S)).\end{aligned}$$

Using the argument in (3.10) for the last term of this expression, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|S_\beta - S\|_{H^{-1}}^2 + (K_\beta(S_\beta) - K_\beta(S), S_\beta - S) &\leq -(K_\beta(S) - K(S), S_\beta - S)\\ &\quad + \frac{\eta}{4}\|f(S_\beta) - f(S)\|_{L^2}^2 + \frac{\|\mathbf{u}\|_{L^\infty}^2}{\eta}\|S_\beta - S\|_{H^{-1}}^2\end{aligned} \tag{4.8}$$

where η is as in inequality (1.5). Next we bound the first term on the right handside of (4.8) by

$$-(K_\beta(S) - K(S), S_\beta - S) \leq C\,\|K_\beta(S) - K(S)\|_{L^\gamma}^\gamma + \frac{C^*}{4}\|S_\beta - S\|_{L^{2+\nu}}^{2+\nu} \tag{4.9}$$

where C^* is as in Lemma 3.1 and we have used the inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \qquad \text{if } \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

We may then 'bury' the second term of (4.9) in the second term on the left handside of (4.8) by Lemma 3.1. Similarly, we can hide the second term of the right handside of (4.8) in the left handside by (1.5) to yield

$$\frac{1}{2}\frac{d}{dt}\|S_\beta - S\|^2_{H^{-1}} + \frac{1}{2}(K_\beta(S_\beta) - K_\beta(S), S_\beta - S)$$
$$\leq C\,\|K_\beta(S) - K(S)\|^\gamma_{L^\gamma} + \frac{\|\mathbf{u}\|^2_{L^\infty}}{\eta}\|S_\beta - S\|^2_{H^{-1}}.$$

According to Hölder's inequality,

$$\|K - K_\beta\|_{L^\gamma[0,1]} \leq \|K - K_\beta\|_{L^\infty[0,1]}$$

and so the previous inequality becomes

$$\frac{d}{dt}\|S_\beta - S\|^2_{H^{-1}} + (K_\beta(S_\beta) - K_\beta(S), S_\beta - S) \;\leq\; C(\beta) + C_1(\mathbf{u})\|S_\beta - S\|^2_{H^{-1}},$$

where $C_1(\mathbf{u}) = \dfrac{2\,\|\mathbf{u}\|^2_{L^\infty}}{\eta}$. We can now apply the Grönwall Lemma to obtain inequality (4.5).

Inequalities (4.6) and (4.7) follow simply as before in Lemma 3.4: inequality (4.6) follows from the Lipschitz property (3.21) of K_β together with (4.5), while inequality (4.7) follows directly from (4.5) and Lemma 3.1. $\square$

The next three results provide smoothness estimates for solutions of the regularized homogeneous equation (4.4) $(Q \equiv 0,\, q \equiv 0)$, and allow us to establish additional estimates similar to those appearing in Section 3. The following identity is used in our first lemma: If $\phi \in C^2(\overline{\Omega} \times [0, T_0])$ and $G(t) := \left\{x \in \overline{\Omega} \mid \phi(x,t) > 0\right\}$, then

$$\frac{d}{dt}\left(\int_{G(t)} \phi(x,t)\,dx\right) = \int_{G(t)} \frac{\partial\phi}{\partial t}(x,t)\,dx\;. \tag{4.10}$$

This identity is verified by parameterizing $G(t)$ and applying the chain rule in combination with Leibnitz's theorem on differentiation of integrals. The boundary terms vanish by observing that $\phi = 0$ on $\partial G(t) \cap \Omega$.

Lemma 4.2 Assume that $k_\beta \geq k$ and that k, f satisfy conditions (1.2) and (1.5). If S_β is the solution to the homogeneous equation (4.4), then there are constants C_1, C_2 (independent of β) such that

$$\left\|\frac{\partial S_\beta}{\partial t}\right\|_{L^\infty(0,T_0,L^1(\Omega))} \leq C_1 + C_2\,\|S^0\|_{W^1_2(\Omega)}\;. \tag{4.11}$$

Proof. We apply the remark (4.10) above to the function $\phi = (S_\beta)_t$, in which case

$$G(t) := \left\{x \in \overline{\Omega} \,\Big|\, \frac{\partial S_\beta}{\partial t}(x,t) > 0\right\}$$

and

$$\frac{d}{dt}\left(\int_{G(t)} \frac{\partial S_\beta}{\partial t}(x,t)\,dx\right) = \int_{G(t)} (S_\beta)_{tt}\,(x,t)\,dx\ .$$

Upon differentiating (4.4a) with respect to t, integrating over $G(t)$ and applying the divergence theorem however, we also obtain

$$\int_{G(t)} (S_\beta)_{tt}\,dx = \int_{\partial G(t)} -(f(S_\beta)\mathbf{u})_t\cdot\mathbf{n} + \frac{\partial}{\partial \mathbf{n}}[K_\beta(S_\beta)_t]\,d\sigma$$

and so

$$\begin{aligned} \frac{d}{dt}\int_\Omega \left[\frac{\partial S_\beta}{\partial t}\right]^+ dx &= \frac{d}{dt}\left(\int_{G(t)} \frac{\partial S_\beta}{\partial t}(x,t)\,dx\right) \\ &= \int_{\partial G(t)} -(f(S_\beta)\mathbf{u})_t\cdot\mathbf{n} + \frac{\partial}{\partial \mathbf{n}}[K_\beta(S_\beta)_t]\,d\sigma \\ &= \int_{\partial G(t)\cap\Omega} -(f(S_\beta)\mathbf{u})_t\cdot\mathbf{n} + \frac{\partial}{\partial \mathbf{n}}[K_\beta(S_\beta)_t]\,d\sigma\ . \end{aligned} \tag{4.12}$$

In the last boundary integral of identity (4.12), we used the homogeneous boundary condition (4.4b) to replace $\partial G(t)$ by $\partial G(t)\cap\Omega$. We next observe that

$$\frac{\partial}{\partial \mathbf{n}}(K_\beta(S_\beta)_t) \le 0, \qquad \text{on } \partial G(t)\cap\Omega. \tag{4.13}$$

This follows from the fact that by the chain rule $K_\beta(S_\beta)_t = k_\beta(S_\beta)\,(S_\beta)_t$ which is positive on $G(t)$ and vanishes on $\partial G(t)\cap\Omega$ with $(S_\beta)_t$. Substituting this back into (4.12), we get the inequality

$$\frac{d}{dt}\int_\Omega \left[\frac{\partial S_\beta}{\partial t}\right]^+ dx \le \int_{\partial G(t)\cap\Omega} -(f(S_\beta)\mathbf{u})_t\cdot\mathbf{n}\ d\sigma.$$

Again, since $S_{\beta t}=0$ on $\partial G(t)\cap\Omega$, we obtain $(f(S_\beta)\mathbf{u})_t = f(S_\beta)\mathbf{u}_t$ on $\partial G(t)\cap\Omega$. Thus the above inequality leads to

$$\frac{d}{dt}\int_\Omega \left[\frac{\partial S_\beta}{\partial t}\right]^+ dx \le \int_{\partial G(t)\cap\Omega} [-f(S_\beta)\,\mathbf{u}_t\cdot\mathbf{n}]d\sigma. \tag{4.14}$$

Using the fact that the boundary of Ω has finite $n-1$ dimensional Lebesgue measure and following with an application of the divergence theorem, we obtain

$$\begin{aligned} \int_{\partial G(t)\cap\Omega} -f(S_\beta)\,\mathbf{u}_t\cdot\mathbf{n}\ d\sigma &\le c + \int_{\partial G(t)} [-f(S_\beta)\,\mathbf{u}_t]\cdot\mathbf{n}\ d\sigma \\ &\le c + \int_{G(t)} \nabla\cdot[-f(S_\beta)\,\mathbf{u}_t]\ dx \end{aligned} \tag{4.15}$$

where

$$c := \int_{\partial G(t)\cap\partial\Omega} |f(S_\beta)\,\mathbf{u}_t|\ \ d\sigma\ .$$

We recall from part ii.) of Proposition 3.2 that

$$|f'| \le c\sqrt{k} \tag{4.16}$$

and we may therefore estimate the integral expression on the right handside of the last inequality of (4.15) by

$$\begin{aligned}\int_{G(t)} \nabla\cdot[\,-f(S_\beta)\,\mathbf{u}_t]\;dx \;&\le\; \int_\Omega |\nabla\cdot\mathbf{u}_t| + |\mathbf{u}_t\cdot\nabla f(S_\beta)|\;\;dx \\ &\le\; c(\mathbf{u})\left(1+\int_\Omega \sqrt{k(S_\beta)}\,|\nabla S_\beta|\,dx\right)\;.\end{aligned} \tag{4.17}$$

If we combine inequalities (4.14)–(4.17), integrate over $[0,t]$ and follow with Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega \left[\frac{\partial S_\beta}{\partial t}\right]^+ dx \;&\le\; c(\mathbf{u})\left\|\sqrt{k(S_\beta)}\nabla S_\beta\right\|_{L^1(L^1)} + \int_{G_+(0)} S_{\beta t}(x,0)\,dx \\ &\le\; c(\mathbf{u})\left\|\sqrt{k(S_\beta)}\nabla S_\beta\right\|_{L^2(L^2)} + \int_{G_+(0)} S_{\beta t}(x,0)\,dx\;.\end{aligned}$$

According to Theorem 3.7, the first expression on the right handside is bounded by $(C\;T_0 + \|S^0\|^2_{L^2})^{\frac{1}{2}}$, while the last term may be expressed as:

$$\int_{G_+(0)} S_{\beta t}(x,0)\,dx = \int_{G_+(0)} (\,-\nabla\cdot f(S^0)\mathbf{u}(x,0) + \Delta K_\beta(S^0))dx$$

which is bounded by a constant multiple of $\|S^0\|_{W_2^1}$, independent of β. Accordingly, the term $\|S^0\|^2_{L^2}$ from Theorem 3.7 may be estimated by $\|S^0\|_{W_2^1}$ by the Sobolev embedding theorem for $n\le 3$.

A similar proof holds for estimating $(S_{\beta t})^-$, which gives the Lemma with a constant depending on T_0, k, f, $\mathbf{u}$, and the initial condition S_0, but independent of β. □

Lemma 4.3 Suppose k,f satisfy conditions (1.2) and (1.5). There is a positive constant $C = C(k,f,\mathbf{u})$ so that if S_β is the solution to the homgeneous equation (4.4), then for all $\beta > 0$

$$\left\|\sqrt{k_\beta(S_\beta)}\,\frac{\partial S_\beta}{\partial t}\right\|^2_{L^2(L^2)} + \|\nabla K_\beta(S_\beta)\|^2_{L^\infty(L^2)} \;\;\le\;\; C + \left\|\nabla K_\beta(S^0)\right\|^2_{L^2}. \tag{4.18}$$

Proof. If one multiplies equation (4.4a) by $K_\beta(S_\beta)_t$ and integrates over Ω, then

$$\Big(\,S_{\beta t}, K_\beta(S_\beta)_t\,\Big)_\Omega + \Big(\,\nabla\cdot[f(S_\beta)\mathbf{u} - \nabla K_\beta(S_\beta)], K_\beta(S_\beta)_t\,\Big)_\Omega = 0,$$

and so integration by parts together with the boundary condition (4.4b) will yield

$$\Big(S_{\beta t}\;,\;K_\beta(S_\beta)_t\Big) + \Big(\nabla K(S_\beta)\;,\;(\nabla K_\beta(S_\beta))_t\Big) = \Big(f(S_\beta)\mathbf{u}\;,\;(\nabla K_\beta(S_\beta))_t\Big).$$

Hence,

$$\begin{aligned}&\frac{1}{2}\left\|\sqrt{k_\beta(S_\beta)}\;(S_\beta)_t\right\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}\|\nabla K_\beta(S_\beta)\|^2_{L^2} \\ &\qquad = \Big(f(S_\beta)\mathbf{u},(\nabla K_\beta(S_\beta))_t\Big) \\ &\qquad = -\Big((f(S_\beta)\mathbf{u})_t,\nabla K_\beta(S_\beta)\Big) + \frac{d}{dt}(f(S_\beta)\mathbf{u},\nabla K_\beta(S_\beta)),\end{aligned} \tag{4.19}$$

where we have used the product rule in the last equation. We estimate the first term on the right handside of this equation by differentiating the product, and in succession apply Cauchy–Schwarz, the triangle inequality, the fact from part ii.) of Proposition 3.2 that $|f'| \le c\sqrt{k}$, and the arithmetic-geometric mean inequality to obtain

$$\begin{aligned}\Big|\Big((f(S_\beta)\mathbf{u})_t, \nabla K_\beta(S_\beta)\Big)\Big| &\le C\left(\left\|\sqrt{k_\beta(S_\beta)}\ (S_\beta)_t\right\|_{L^2} + \|f(S_\beta)\mathbf{u}_t\|_{L^2}\right)\ \|\nabla K_\beta(S_\beta)\|_{L^2} \\ &\le \tfrac{1}{2}\left\|\sqrt{k_\beta(S_\beta)}\ (S_\beta)_t\right\|_{L^2}^2 + C\,\|\nabla K_\beta(S_\beta)\|_{L^2}^2 + C_1\ .\end{aligned}$$

We remind the reader that our constants depend only upon $\mathbf{u}$, f, and k, but may be changing from line to line. Substituting this back into inequality (4.19) and then subtracting the first term on the right handside from the like term on the left, we obtain

$$\begin{aligned}&\left\|\sqrt{k_\beta(S_\beta)}\ (S_\beta)_t\right\|_{L^2}^2 + \tfrac{d}{dt}\,\|\nabla K_\beta(S_\beta)\|_{L^2}^2 \\ &\qquad\le C\,\|\nabla K_\beta(S_\beta)\|_{L^2}^2 + C_1 + \tfrac{d}{dt}\Big(f(S_\beta)\mathbf{u}, \nabla K_\beta(S_\beta)\Big).\end{aligned}$$

Consequently, an application of Grönwall's lemma leads to

$$\begin{aligned}&\left\|\sqrt{k_\beta(S_\beta)}\ (S_\beta)_t\right\|_{L^2(L^2)}^2 + \left\|\nabla K_\beta(S_\beta)\right\|_{L^\infty(L^2)}^2 \\ &\qquad\le C_1\,T_0 + C_2 \sup_{0\le t\le T_0}\Big|\Big(f(S_\beta)\mathbf{u}, \nabla K_\beta(S_\beta)\Big)\Big| + \left\|\nabla K_\beta(S^0)\right\|_{L^2}^2 .\end{aligned} \tag{4.20}$$

The arithmetic-geometric mean inequality may be applied again to the inner product on the right handside of (4.20) as follows

$$\Big|\Big(f(S_\beta)\mathbf{u}, \nabla K_\beta(S_\beta)\Big)\Big| \le \frac{C_2}{2}\|f(S_\beta)\mathbf{u}\|_{L^\infty(L^2)}^2 + \frac{1}{2C_2}\|\nabla K_\beta(S_\beta)\|_{L^\infty(L^2)}^2$$

which allows us to incorporate this last term on the right into the corresponding term on the left handside of (4.20) to complete the proof of the Lemma. □

Theorem 4.4 Assume that $k_\beta \ge k$, that the coefficient k satisfies condition (1.2), and the pair k, f satisfies the condition (1.5). Further assume that S_β solves the homogeneous, regularized equation (4.4). If we define

$$m(\beta) := \inf_{0\le s\le 1} k_\beta(s)\ , \tag{4.21}$$

then there is a constant C, independent of β, so that

$$\left\|\frac{\partial S_\beta}{\partial t}\right\|_{L^\gamma(L^\gamma)} \le C\ m(\beta)^{-\frac{1}{2+\nu}} \tag{4.22}$$

and hence

$$\left\|\frac{\partial S_\beta}{\partial t} + \nabla\cdot f(S_\beta)\mathbf{u}\right\|_{L^\gamma(L^\gamma)} \le C\ m(\beta)^{-\frac{1}{2+\nu}} \tag{4.23}$$

where $\gamma = \frac{2+\nu}{1+\nu}$ is defined in (4.3).

Proof. Lemma 4.3 gives

$$m(\beta) \left\| \frac{\partial S_\beta}{\partial t} \right\|^2_{L^2(L^2)} \leq C. \tag{4.24}$$

This is (4.22) if $\nu = 0$. If $\nu > 0$, then $\frac{1}{2} < \frac{1}{\gamma} < 1$ and so setting $\theta := \frac{2}{2+\nu}$, we have

$$\frac{1}{\gamma} = \theta \, \frac{1}{2} + (1-\theta)\, 1.$$

With this value of θ, a corollary of Hölder's inequality states that

$$\|\phi\|_{L^\gamma} \leq \|\phi\|_{L^1}^{1-\theta} \|\phi\|_{L^2}^{\theta}$$

from which we obtain

$$\begin{aligned} \|(S_\beta)_t\|^\gamma_{L^\gamma} &\leq \|(S_\beta)_t\|_{L^1}^{\frac{\nu}{1+\nu}} \, \|(S_\beta)_t\|_{L^2}^{\frac{2}{1+\nu}} \\ &\leq C^{\frac{\nu}{1+\nu}} \left(\int_\Omega |(S_\beta)_t|^2 dx \right)^{\frac{1}{1+\nu}} . \end{aligned}$$

where we have used Lemma 4.2 for the L^1 estimate. If we integrate this inequality over the interval $[0, T_0]$ and apply Hölder's inequality, we see that

$$\|(S_\beta)_t\|^\gamma_{L^\gamma(L^\gamma)} \leq C^{\frac{\nu}{1+\nu}} \, \|(S_\beta)_t\|^{\frac{2}{1+\nu}}_{L^2(L^2)}.$$

Substituting the estimate (4.24) into this last inequality gives the estimate (4.22) of the Theorem.

Inequality (4.23) is derived from (4.22) by applying Holder's inequality and the product rule

$$\nabla \cdot f(S_\beta)\mathbf{u} = f'(S_\beta)\nabla S_\beta \cdot \mathbf{u} + f(S_\beta)\nabla \cdot \mathbf{u}$$

to estimate the gradient term as follows:

$$\|\nabla \cdot f(S_\beta)\mathbf{u}\|_{L^\gamma(L^\gamma)} \leq \|\nabla \cdot f(S_\beta)\mathbf{u}\|_{L^2(L^2)} \leq \left\| \sqrt{k_\beta(S_\beta)} \, \mathbf{u} \cdot \nabla S_\beta \right\|_{L^2(L^2)} + \|f(S_\beta)\nabla \cdot \mathbf{u}\|_{L^2(L^2)}$$

where part ii.) of Proposition 3.2 was used to estimate f' by $\sqrt{k_\beta}$. By applying inequality (3.26) of Theorem 3.7, the proof is complete. □

Remark 4.5 For our last result, we choose a specific regularization, i.e. a specific k_β defined by

$$k_\beta(s) := \max\left(k(s), \beta^\nu\right) \tag{4.25}$$

in which case $m(\beta) \geq \beta^\nu$ and $\|K_\beta - K\|_{L^\infty} \leq 2C\,\beta^{1+\nu}$ for β sufficiently small. Consequently, the constant $C(\beta)$ defined in Theorem 4.1 satisfies

$$C(\beta) \leq C\,\beta^{2+\nu} .$$

Using this particular family k_β (or even an appropriate C^∞ family majorizing this family), we may establish estimates similar to inequality (4.5) of Theorem 4.1 as well as regularity results in terms of $\|\nabla(K(S_\beta) - K(S))\|_{L^2(L^2)}$.

Theorem 4.6 Assume that $k_\beta \geq k$ is given by (4.25), that the coefficient k satisfies condition (1.2), and the pair k, f satisfies the condition (1.5). Further assume that S_β solves the homogeneous, regularized equation (4.4). If we define $\delta(\beta) := \beta^{\min(1,\nu)}$, then

$$\sup_{0\leq t\leq T_0} (K(S_\beta) - K(S), S_\beta - S) + \|\nabla(K(S_\beta) - K(S))\|^2_{L^2(L^2)} \leq C\,\delta(\beta). \tag{4.26}$$

$$\|S_\beta - S\|^{2+\nu}_{L^\infty(L^{2+\nu})} \leq C\,\delta(\beta). \tag{4.27}$$

$$\|K(S_\beta) - K(S)\|^2_{L^2(0,T_0,H^1)} \leq C\,\delta(\beta). \tag{4.28}$$

Proof. Subtract equations (3.1a) from (4.4a), and integrate against $K(S_\beta) - K(S)$ over Ω to get

$$\Big((S_\beta - S)_t, K(S_\beta) - K(S)\Big) + \Big(\nabla\cdot(f(S_\beta) - f(S))\mathbf{u}, K(S_\beta) - K(S)\Big) - \Big(\Delta(K_\beta(S_\beta) - K(S)), K(S_\beta) - K(S)\Big) = 0$$

We can rewrite this identity, using the product rule and the boundary condition (4.4b) to obtain

$$\begin{aligned}
&\frac{d}{dt}(S_\beta - S, K(S_\beta) - K(S)) + \|\nabla(K(S_\beta) - K(S))\|^2_{L^2} \\
&\quad= ((f(S_\beta) - f(S))\mathbf{u}, \nabla(K(S_\beta) - K(S))) \\
&\qquad+ (S_\beta - S, (K(S_\beta) - K(S))_t) \\
&\qquad+ (\nabla(K(S_\beta) - K_\beta(S_\beta)), \nabla(K(S_\beta) - K(S))) \\
&\quad= \mathrm{I} + \mathrm{II} + \mathrm{III}\,.
\end{aligned} \tag{4.29}$$

We treat individually each term on the right handside of (4.29).

The *first term* I may be bounded as follows

$$\begin{aligned}
&\Big((f(S_\beta) - f(S))\mathbf{u}, \nabla(K(S_\beta) - K(S))\Big) \\
&\qquad\leq \|(f(S_\beta) - f(S))\mathbf{u}\|^2_{L^2} + \tfrac{1}{4}\|\nabla(K(S_\beta) - K(S))\|^2_{L^2}.
\end{aligned} \tag{4.30}$$

where the rightmost term of (4.30) will be hidden in the left handside of (4.29) to be performed later after an integration over $[0, T_0]$. The remaining term on the right handside of (4.30) is estimated using the fact that f is C^1 (thus Lipschitz), followed by Hölder's inequality,

$$\|(f(S_\beta) - f(S))\mathbf{u}\|^2_{L^2} \leq \|\mathbf{u}\|^2_{L^\infty}\|f'\|^2_{L^\infty}\|S_\beta - S\|^2_{L^2} \leq C\,\|S_\beta - S\|^2_{L^{2+\nu}}.$$

By applying inequality (4.7) of Theorem 4.1 and integrating over $[0, T_0]$, we then obtain that

$$\|(f(S_\beta) - f(S))\mathbf{u}\|^2_{L^2(L^2)} \;\leq\; C\,\beta^2\,. \tag{4.31}$$

The *second term* II of (4.29) can be bounded by first applying Hölder's inequality over $\Omega\times[0,T_0]$ and the fact that $1\le\gamma\le 2$ to obtain by Theorem 4.1 and Lemma 4.3 (since $\sqrt{k}$ is bounded):

$$\begin{aligned} \textstyle\int_0^{T_0} \Big(S_\beta - S, (K(S_\beta)-K(S))_t \Big)_\Omega \, d\tau & \\ &\le \ \|S_\beta - S\|_{L^{2+\nu}(L^{2+\nu})} \|(K(S_\beta)-K(S))_t\|_{L^\gamma(L^\gamma)} \\ &\le \ c\, C(\beta)^{\frac{1}{2+\nu}} \ \|(K(S_\beta)-K(S))_t\|_{L^\infty(L^2)} \ \le \ c\,\beta \,. \end{aligned} \tag{4.32}$$

The *third term* III of equation (4.29) is estimated according to

$$\begin{aligned} \Big(\nabla(K(S_\beta)-K_\beta(S_\beta)), \nabla(K(S_\beta)-K(S)) \Big)_\Omega & \\ &\le \ \|\nabla(K(S_\beta)-K_\beta(S_\beta)\|_{L^2}^2 + \tfrac14 \|\nabla(K(S_\beta)-K(S))\|_{L^2}^2 \,. \end{aligned} \tag{4.33}$$

Again, after integration over $[0,T_0]$, we will hide the leftmost term on the right handside of (4.33) in the left handside of (4.29), and treat the first term as follows: Note by the definition of k_β in (4.25) and its properties that

$$|k_\beta(\xi)-k(\xi)| \ \le \ \beta^\nu \, \chi_{\{k(\xi)<\beta^\nu\}}(\xi) \ \le \ \beta^{\frac{\nu}{2}} \sqrt{k_\beta(\xi)} \,.$$

and so

$$|\nabla(K(S_\beta)-K_\beta(S_\beta))| = |k_\beta(S_\beta)-k(S_\beta)|\,|\nabla S_\beta| \le \beta^{\frac{\nu}{2}} \sqrt{k_\beta(S_\beta)}\,|\nabla S_\beta|.$$

Consequently, it follows from Theorem 3.7 that

$$\begin{aligned} \|\nabla(K(S_\beta)-K_\beta(S_\beta))\|_{L^2(L^2)} \ &\le \ \beta^{\frac{\nu}{2}} \Big\| \sqrt{k_\beta(S_\beta)}\,|\nabla S_\beta| \Big\|_{L^2(L^2)} \\ &\le \ C\,(T_0 + \|S_0\|_{L^2})\,\beta^{\frac{\nu}{2}}. \end{aligned} \tag{4.34}$$

Finally, we complete the proof of estimate (4.26) by integrating identity (4.29) and applying inequalities (4.30)–(4.34) to obtain

$$\begin{aligned} \sup_{0\le t\le T_0} \Big(S_\beta - S, K(S_\beta)-K(S) \Big)_\Omega + \Big\| \nabla\Big(K(S_\beta)-K(S)\Big)\Big\|_{L^2(L^2)}^2 & \\ \le C\left(\beta^2+\beta+\beta^\nu\right) + \frac12 \, \Big\| \nabla\Big(K(S_\beta)-K(S)\Big)\Big\|_{L^2(L^2)}^2 & \end{aligned} \tag{4.35}$$

and follow by carrying the rightmost term over to the left handside as we indicated earlier.

Inequality (4.27) follows immediately from (4.26) as before by using Lemma 3.1. Applying Holder's inequality together with the facts that $(K(b)-K(a))^2 \ \le \ \|k\|_\infty\,(K(b)-K(a))(b-a)$ and inequality (4.26) holds, implies that

$$\|K(S_\beta)-K(S)\|_{L^2(0,T_0,H^1)}^2 \ \le \ \|k\|_\infty \sup_{0\le t\le T_0} \Big(K(S_\beta)-K(S), S_\beta - S\Big)_\Omega + \|\nabla(K(S_\beta)-K(S))\|_{L^2(L^2)}^2$$

and therefore this quantity is bounded by $C\,\delta(\beta)$, which verifies inequality (4.28). □

REFERENCES

1. R.A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
2. J. Bear and A.Verruijt, "Modeling Groundwater Flow and Pollution," D. Reidel Publication Company, Dordrech, Holland, 1987.
3. P. G. Ciarlet, "The Finite Element Methods for Elliptic Problems," North–Holland, 1978.
4. R. Dautray and J–L. Lions, "Mathematical Analysis and Numerical Methods for Science and Technology," Vol. 2, Springer–Verlag, New York, 1984.
5. R.E. Ewing, *Problems arising in the modeling of processes for hydrocarbon recovery,* in "The Mathematics of Reservoir Simulation," Ed. R.E. Ewing, SIAM, Philadelphia, 1983, pp. 3–34.
6. K.B. Fadimba, "Regularization and Numerical Methods for a Class of Porous Medium Equations," Ph.D Thesis, Univ. of South Carolina, 1993.
7. B. H. Gilding and L. A. Peletier, *The Cauchy Problem for an Equation in the Theory of Infiltration,* Arch. Rat. Mech. Analysis **61**(1976), 126–152.
8. D. W. Peaceman, "Fundamentals of Numerical Reservoir Simulation," Elsevier Scientific Publishing Company, New York, 1977.
9. M. E. Rose, *Numerical Methods for Flow through Porous Media–I,* Math. Comp. **40**(1983), 437–467.
10. M. E. Rose, *Numerical Methods for Flow through Porous Media–II,* Comput. Math. with Appls.**6**(1980),99–122.
11. M. E. Rose, "Numerical Methods for a General Class of Porous Medium Equations," Argonne National Laboratory, Argonne, Il,1980.
12. D. L. Smylie, "A Near Optimal–Order Appproximation to a Class of Two–Sided Nonlinear Degenerate Parabolic Partial Differential Equations," Ph.D Thesis, Univ. of Wyoming, 1989.

Institute for Scientific Computation
Texas A&M University
College Station, TX 77843-3257

Department of Mathematics
University of South Carolina
Columbia, SC 29208