ON THE ATOMIC DECOMPOSITION OF H^1
AND INTERPOLATION

ROBERT SHARPLEY[1]

In [1] Coifman used the Fefferman-Stein theory of H^p spaces [4] to decompose the functions of these spaces into basic building blocks (atoms), further clarifying their real variable nature. Coifman and Weiss have provided a comprehensive treatment of these ideas and many applications to harmonic analysis in [2]. In this note, we use the nontangential maximal function Nf to give an elementary proof of the decomposition of H^1 functions on the line and then characterize the Peetre K-functional for H^1 and L^∞ in terms of Nf.

Let u be the harmonic extension [5] of f to the upper half plane $\mathbf{R}^2_+$. For $x \in \mathbf{R}$, denote by $\Gamma_2 = \{(z,y) \in \mathbf{R}^2_+ : |x - z| \leq y\}$ the cone with vertex at x. The nontangential maximal function of f is defined by $Nf(x) = \sup\{|u(z,y)| : (z,y) \in \Gamma_x\}$. We define the (real) H^1 norm of f to be the standard H^1 norm of $u + iv$, where v is the harmonic conjugate of u which satisfies $v(0) = 0$. A classical result of Hardy and Littlewood asserts that $\|Nf\|_{L^1} \leq c\|f\|_{H^1}$. For an interval I an H^1-atom is any function a_I such that $\int a_I = 0$ and $|a_I| \leq |I|^{-1}\chi_I$ a.e.

PROPOSITION.[2] *Suppose u is harmonic on an open square S and continuous on $\overline{S}$. Then its average on ∂S equals the average over the two diagonals.*

PROOF. By dilating to a subcube of S and then expanding back, we may assume that both u and its harmonic conjugate v are continuous on $\overline{S}$. Now S is composed of four $15°$ right triangles with common vertex the center of S. Let T be the lower triangle and denote its edges by L, B and R, where B is the hypotenuse. Applying Cauchy's theorem to $u + iv$ on T and taking real parts of the integrals gives

$$0 = \oint_{\partial T} u\, dx - v\, dy = \int_B u - \frac{1}{\sqrt{2}}\left(\int_{R+L} u + \int_R v - \int_L v\right).$$

Using rotations and symmetry, applying this argument to the three remaining subtriangles of S, and summing the resulting equations, we see that the terms involving v cancel and we are left with our stated result. □

THEOREM. *If $Nf \in L^1$, then we may write $f = \sum_j \lambda_j a_j$ so that the a_j's are atoms and the coefficients λ_j satisfy*

(1)
$$\sum_j |\lambda_j| \leq 42\|Nf\|_{L^1}.$$

PROOF. Since u is continuous in $\mathbf{R}^2_+$, then $E_k = \{x: Nf(x) > 2^k\}$ is open in $\mathbf{R}$. Let $I(f) = \int_I f\,dx/|I|$ and define F_k as the complement in $\mathbf{R}$ of E_k. We write E_k as the disjoint union of its collection $\mathcal{C}_k$ of open components and then decompose f as a sum, $f = g_k + h_k$, where

$$(2) \qquad g_k = \sum_{I \in \mathcal{C}_k} [f - I(f)]\chi_I, \qquad h_k = f\chi_{F_k} + \sum_{I \in \mathcal{C}_k} I(f)\chi_I.$$

We claim that $|h_k Z| \le 7 \times 2^k$ a.e. Clearly, the estimate holds on F_k since $Nf \le 2^k$ there and $|f| \le Nf$ a.e. For the remaining set E_k, we fix an interval I in $\mathcal{C}_k$ and show that

$$(3) \qquad\qquad\qquad |I(f)| \le 7 \times 2^k.$$

Let S_ε be the open square $I \times (\varepsilon, |I| + \varepsilon)$ in $\mathbf{R}^2_+$. By the Proposition and letting $\varepsilon \downarrow 0$, $I(f)$ is seen to equal four times the average of u over the union of the two main diagonals less the sum of its averages over the three remaining sides. But the endpoints of I belong to F_k, so the diagonals, sides and top of S all belong to the "good" set for u, namely $\Gamma = \{(z,y) \in \Gamma_x: x \in F_k\}$. The definitions of F_k and Nf imply that u is bounded by 2^k on Γ which establishes (3).

Following Coifman [1] and Herz [6f], the atoms are defined by

$$(4) \qquad\qquad a_I \lambda_I^{-1}(g_k - g_{k+1})\chi_I, \qquad \lambda_I = 21 \times 2^k|I|,$$

for each $I \in \mathcal{C}_k$ and all integers k. By telescoping and using both that $g_k - g_{k+1} = h_{k+1} - h_k$ and that g_{k+1} is supported in $E_{k+1} \subset E_k$, it follows that

$$f = \sum_{k=\infty}^{\infty} (g_k - g_{k+1}) = \sum_k \sum_{I \in \mathcal{C}_k} \lambda_I a_I.$$

Each a_I is an atom since it is supported in I and the estimate $\|a_I\|_\infty \le |I|^{-1}$ follows from our L^∞ estimate on the h_k,

$$\|g_k - g_{k+1}\|_\infty = \|h_{k=1} - h_k\|_\infty \le 7(2^{k+1} + 2^k) = 21 \times 2^k.$$

To see that a_I has mean value zero, it suffices to write it in the form

$$a_I = \lambda_I^{-1}\left([f - I(f)]\chi_I - \sum_{\substack{J \in \mathcal{C}_{k+1} \\ J \subset I}} [f - J(f)]\chi_J \right).$$

To establish inequality (1) (subject to relabeling) we use

$$(5) \qquad \sum_k \sum_{I \in \mathcal{C}_k} |\lambda_I| + 21\sum_k 2^k \sum_{I \in \mathcal{C}_k} |I| = 21\sum_k 2^k|E_k| = 21\sum_k (2^{k+1} - 2^k)|E_k|.$$

Indeed by (5), summation by parts, and the fact $Nf > 2^k$ on E_k, we have

$$(6) \qquad\qquad \sum_k \sum_{I \in \mathcal{C}_k} |\lambda_I| \le 42\sum_k 2^k|E_k \backslash E_{k+1}| \le 42 \int Nf(x)\,dx. \quad \square$$

Fefferman, Rivière and Sagher [3] estimated the K-functional

$$K(f,t) = \inf\{\|g\|_{H^1} + t\|h\|_{L^\infty}: g \in H^1, h \in L^\infty, f = g + h\}$$

in terms of the "grand maximal" operator to describe interpolation spaces for the pair. We provide a description in terms of Nf. Let g^* denote the decreasing rearrangement of $|g|$.

COROLLARY (OF THE PROOF OF THE THEOREM). *If f belongs to* $H^1 + L^\infty$, *then*

$$(7) \qquad K(f, t) \sim \int_0^t (Nf)^*(s)\, ds, \qquad t > 0.$$

PROOF. The subadditivity of the integral operator in (7) implies that it is dominated by $K(f, t)$. To establish the opposite estimate, we fix $t > 0$ and select an integer j so that $2^{j-1} < (Nf)^*(t) \le 2^j$. From the constructions in (2), we see

$$(8) \qquad g_j = \sum_{k=j}^\infty (g_k - g_{k+1}) = \sum_{k=j}^\infty \left(\sum_{I \in \mathcal{C}_k} \lambda_I a_I \right).$$

The estimate $\|h_j\|_\infty \le 14(Nf)^*(t)$ follows by our selection of the index j, while

$$(9) \qquad \|g_j\|_{H^1} \le 42c \int_{E_j} Nf(x)\, dx \le 42c \int_0^t (Nf)^*(s)\, ds$$

is derived as in (5)–(6) using the identity (8). Combining these estimates completes the proof. □

Minor modifications using p-atoms permit extension of these results to H^p spaces $(\frac{1}{2} < p < 1)$ on $\mathbf{R}$. Beginning with Nf and using classical techniques (theorems of Spanne-Stein and Hardy-Littlewood), these results provide a simplified approach to the various descriptions of $H^p(\mathbf{R})$ (duality, grand maximal operator). By conformally mapping the unit disc onto $\mathbf{R}_+^2$ and estimating the appropriate integrals obtained from the Proposition, one obtains the expected results for the circle. Exploiting a Fourier analytical technique of Calderón, Wilson has given a proof of the atomic decomposition into L^2 atoms for higher dimensions in [8], while the condition $N(u + iv) \in L^1$ is required in [7]. Finally, the author extends his thanks to Colin Bennett and Guido Weiss for valuable discussions related to this paper.

REFERENCES

1. R. R. Coifman, *A real variable characterization of H^p*, Studia Math. **51** (1974), 269–274.
2. R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their uses in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.
3. C. Fefferman, N. M. Rivière and Y. Sagher, *Interpolation between H^p space: The real method*, Trans. Amer. Math. Soc. **191** (1974), 75–81.
4. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.
5. J. B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
6. C. Herz, *H^p spaces of martingales*, $0 < p \le 1$, Z. Wahrsch. Verw. Gebiete **28** (1974), 189–205.
7. J. M. Wilson, *A simple proof of the atomic decomposition for $H^p(\mathbf{R}^n)$*, $0 < p \le 1$, Studia Math. **74** (1982), 25–33.
8. ____, *On the atomic decomposition for Hardy spaces*, Pacific J. Math. **116** (1985), 201–207.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SOUTH CAROLINA 29208