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INTERPOLATION OF H^1 AND H^∞

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The interpolation spaces for H^1 and H^∞ are characterized as the Hardy spaces of the interpolation spaces for L^1 and L^∞. This description is provided by recent work of Peter Jones on constructive solutions of $\bar{\partial}$ problems and of Brudnyǐ - Krugljak in the general theory of interpolation.

A Banach space X is called an interpolation space of a Banach couple (X_0,X_1) if each linear operator T, whose restriction to X_i is a bounded operator from X_i into itself $(i=0,1)$, is also a bounded operator on X. In [4] Calderón characterized the interpolation spaces of L^1 and L^∞ as the spaces X of measurable functions whose norms satisfy a rearrangement condition; specifically, there must exist a constant $c > 0$ so that

(1) $$f \in X \text{ and } g \prec f \rightarrow \|g\|_X \leq c\, \|f\|_X .$$

Here $g \prec f$ (the Hardy-Littlewood-Polya preorder) means

$$\int_0^t g^*(s)ds \leq \int_0^t f^*(s)ds \quad , \text{all } t > 0$$

and g^* denotes the decreasing rearrangement of $|g|$ [4]. We call such spaces X rearrangement-invariant function spaces. In this brief note we will provide a description of the interpolation spaces for the Banach couple of Hardy spaces (H^1,H^∞). For simplicity we work on R and the upper half plane R^2_+, but similar results hold for T and the disc. The Hardy space

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$H(X)$ of a rearrangement-invariant function space X over R is defined to be the collection of functions f in X which have analytic extensions into the upper half plane and whose norm is given by

(2) $$||f||_{H(X)} = || \, |f| \, ||_X \ .$$

The standard notation of H^p $(=H^p(R))$ will be used for $H(L^p)$. The main ingredients of the proof are identification of the K-functional for (H^1,H^∞) by Peter Jones [5], using rather deep constructive results for $\bar\partial$ problems, together with recent work in general interpolation theory by Brudnyĭ and Krugljak [3].

THEOREM. <u>A necessary and sufficient condition for a space Y to be an interpolation space for the pair (H^1,H^∞) is that Y be equal (with equivalent norms) to a Hardy space $H(X)$ for some interpolation space X of the pair (L^1,L^∞) (i.e., for some rearrangement-invariant function space X).</u>

Proof. From the proof of Theorem 3 in Jones [5] it follows immediately that the Peetre K-functional (see [3], [7] page 261) for the pair (H^1,H^∞) can be estimated by

(3) $$c_1 \ K(f,t) \le \int_0^t (Nf)^*(s)ds \le c_2 \ K(f,t) \qquad , \text{all } t > 0$$

for some fixed positive constants c_i $(i=1,2)$. Here Nf is the nontangential maximal function of f in R^2_+ ; i.e. if F is the harmonic extension of f into R^2_+ , then Nf is defined by

$$Nf\ (x) = \sup\{|F(t,y)| : (t,y) \in R^2_+, \ |x-t| \le y\} \ .$$

For our purposes, a slight improvement of (3) is required, namely

(4) $$c_1 \ K(f,t) \le \int_0^t f^*(s)ds \le c_2 \ K(f,t) \qquad , \text{all } \ t > 0 \ .$$

The right hand inequality follows immediately from (3) since $|f| \le Nf$ a.e. The inequality is evident directly as well since $\int_0^t f^*$ is a subadditive functional of f, $\int_0^t g^* \le ||g||_{L^1} = ||g||_{H^1}$ and $\int_0^t h^* \le t||h||_{L^\infty} = t||h||_{H^\infty}$.

Hence

$$\int_0^t f^*(s)ds \leq \inf_{f=g+h} \{||g||_{H^1} + t||h||_{H^\infty}\} = K(f,t) .$$

For the left hand inequality in (4), let F denote the analytic extension of f into the upper half plane. Factor F as BG^2 where B is a Blaschke product and G is a zero-free analytic function in R_+^2 . Let g be the function on R of boundary values of G, then $Nf \leq (Ng)^2$. Hence

$$\begin{aligned} \int_0^t (Nf)^*(s)ds &\leq \int_0^t (Ng)^*(s)^2 ds \\ &\leq c\int_0^t (Mg)^*(s)^2 ds \end{aligned} \tag{5}$$

since Ng is no larger than a constant multiple of the Hardy-Littlewood maximal function Mg (see page 197 of [8]). In addition, Herz's inequality (see, for example,[1]) states that $(Mg)^*(s) \leq \frac{5}{s} \int_0^s g^*(r)dr$, so using the specialized Hardy inequality

$$\int_0^t [\int_0^s g^*(r)dr/s]^2 ds \leq 4 \quad \int_0^t g^*(s)^2 ds$$

(obtained from integration by parts), we obtain

$$\int_0^t (Mg)^*(s)^2 ds \leq c \int_0^t g^*(s)^2 ds.$$

Together with (5) and the fact that $|g|^2 = |f|$ a.e. this shows that for all $t > 0$

$$\int_0^t (Nf)^*(s)ds \leq c \int_0^t f^*(s)ds$$

for each f belonging to $H^1 + H^\infty$. This inequality together with Jones' inequality (3) establishes (4).

Suppose now that X is a rearrangement-invariant function space and $H(X)$ is its corresponding Hardy space. If T is a bounded linear operator on both H^1 and H^∞, then obviously

$$K(Tf,t) \leq A\ K(f,t) \qquad \text{all} \quad t > 0$$

where A is the maximum of the two operator norms. If the estimate (4) is applied to each side of the last inequality, then Calderón's result (1) shows that $H(X)$ is an interpolation space. Conversely, let Y be an interpolation space for the pair (H^1, H^∞). Jones has shown [6] that if both f and g belong to $H^1 + H^\infty$ and $g \prec f$, then there exists a linear operator T such that $Tf = g$ and T is bounded on both H^1 and H^∞. Applying Corollary 3 of [3], Y is equal to the space $K_\phi(H^1, H^\infty)$ and

$$||f||_Y = \Phi(K(f,\cdot))$$

where Φ is the function norm of some interpolation space for the pair $(L^\infty, L^\infty_{1/t})$. It follows from (1) that $\Phi(\int_0^{(\cdot)} g^*(s)ds) =: ||g||_X$ is a rearrangement-invariant function norm. Hence by the estimates in (4),

$$c_1 ||f||_Y \leq ||f||_X \leq c_2 \ ||f||_Y \quad .$$

This result may be rephased in terms of real Hardy spaces in an obvious way. Recall that ReH^p is the space of functions in L^p whose Hilbert transforms also belong to L^p. For $1 < p < \infty$, Riesz's theorem shows that ReH^p is L^p with an equivalent norm. In general, for a rearrangement-invariant function space X the real Hardy space of X is defined by

$$\text{Re } H(X) = \{f \in X : Hf \in X\}$$

with norm

$$||f||_{ReH(X)} = ||f||_X + ||Hf||_X \ .$$

Here Hf denotes the Hilbert transform of f.

COROLLARY. <u>The K-functional for the pair</u> (ReH^1, ReH^∞) <u>is equivalent within constants to the expression</u>

$$\int_0^t f^*(s)ds + \int_0^t (Hf)^*(s)ds \quad . \tag{6}$$

<u>The collection of interpolation spaces for this pair are precisely the real</u>

Hardy spaces of the interpolation spaces of (L^1, L^∞). A necessary and sufficient condition for a rearrangement-invariant Banach function space X to be an interpolation space for (ReH^1, ReH^∞) is that the Boyd indices of X satisfy $0 < \beta_X \leq \alpha_X < 1$.

Proof. Only the last statement requires verification, but in view of (6), this is precisely the content of [2].

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