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CONE CONDITIONS AND THE MODULUS OF CONTINUITY

Robert C. Sharpley[1]

Abstract. Cone conditions on domains introduced by several authors are shown to be equivalent and are used to give inequalities for moduli of continuity in $L_p(\Omega)$.

In the literature several regularity conditions (I)-(IV) on the boundary of an open set Ω which seem rather close to one another have been introduced to study smoothness of functions and extension operators. In Theorem 1 we, in fact, show that these conditions are all equivalent. In practice, property (I) is usually the easiest to check for a given domain. Domains satisfying property (II) were called *LG domains* by Johnen and Scherer, while (III) specifies domains with *minimally smooth boundaries*. In Proposition 2 we give a somewhat less technical proof of the result of Johnen and Scherer that the r-th order modulus of continuity is related to the K functional for the spaces $L_p(\Omega)$ and $W_p^r(\Omega)$, the Sobolev space of order r. The remainder of the paper deals with the Johnen-Scherer result and extension operators. The argument to prove inequality (7) below is essentially repeated from [4] but is included so that one can see how the result goes in R^n without the hindering features of working on a domain.

We shall use the standard multiindex notation, e.g. letting

$$D^{\nu}f = D_{e_1}^{\nu_1}\cdots D_{e_n}^{\nu_n}f$$

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$$t^\nu = t_1^{\nu_1} \ldots t_n^{\nu_n}$$

and

$$|\nu| = \nu_1 + \ldots + \nu_n$$

where $\nu = (\nu_1, \ldots, \nu_n)$ and $t = (t_1, \ldots, t_n)$. We mean by a cube, an n dimensional cube with edges parallel to the coordinate axes. Finally we say that a domain Ω is a special Lipschitz domain if after a suitable rotation, its boundary is the graph of a Lip 1 map ϕ: $\mathbf{R}^{n-1} \to \mathbf{R}$, i.e.

$$\partial\Omega' = \{(u, \phi(u)):\ u \in \mathbf{R}^{n-1}\}$$

where $\Omega' = \{(u,v):\ u \in \mathbf{R}^{n-1},\ v > \phi(u)\}$ is a rotation of Ω. The bound for the domain is defined to be the infinum of the Lip 1 norms of all such ϕ.

To begin we list the four conditions on a domain which we intend to investigate.

(I) There exists a countable collection of open sets $\{U_j\}$, a corresponding collection of cones $\{C_j\}$ (all congruent to a fixed finite cone C), a real number $\varepsilon > 0$, and a positive integer N with the properties

i) $\partial\Omega \subset \bigcup_j U_j^\varepsilon$, where $U^\varepsilon := \{x:\ B_\varepsilon(x) \subset U\}$,

ii) $x + C_j \subset \Omega$, for each $x \in U_j \cap \Omega$

iii) $\sum_j \chi_{U_j} \leq N$.

(II) (Johnen-Scherer [6]) Same as (I) except that the collection $\{U_j\}$ must be finite (in which case part iii) is redundant).

(III) (Stein, [8, p. 189]) There exists a countable collection of open sets $\{U_j\}$, a corresponding collection of special Lipschitz domains $\{\Omega_j\}$ with uniform upper bound M, a real number $\varepsilon > 0$, and a positive integer N such that

i) if $x \in \partial\Omega$, then $B_\varepsilon(x) \subset U_j$ for some j

ii) $U_j \cap \Omega = U_j \cap \Omega_j$, all j

iii) no point in $\mathbf{R}^n$ is contained in more than N of the U_j's.

It is clear that i) and iii) are, respectively, the same as (I) i) and iii).

(IV) (DeVore-Sharpley [5]) Same as (III) except that iii) is replaced by iii) $B_\varepsilon(x)$ intersects at most N of the U_j's.

Obviously, if Ω were bounded, this condition would follow immediately (with a possibly different N) by compactness.

THEOREM 1. Let Ω be an open subset of $\mathbf{R}^n$ $(n \geq 2)$, then the conditions (I)-(IV) on Ω are all equivalent.

PROOF. We show (I) → (II) → (III) → (IV) → (I). Assuming Ω satisfies (I) we only need to show that $\{U_j\}$ can be taken as a finite collection in order that (II) be satisfied. Observe first of all that by narrowing the vertex angle of the common cone C somewhat, we can select a finite number of cones $C'_1, \ldots, C'_L$ all congruent to a fixed cone $C' \subset C$ such that for each j there is a cone C'_ℓ $(1 \leq \ell \leq L)$ such that $C'_\ell \subset C_j$. This follows since the unit sphere S^{n-1} of $\mathbf{R}^n$ is compact and we may identify the cones with open balls in S^{n-1} (all with the same radius) and the axes of the cones with the corresponding centers. Now let $I(\ell)$ be the index set of all j such that $C'_\ell \subset C_j$ and set

$$U'_\ell := \bigcup_{j \in I(\ell)} U_j ,$$

then these open sets and cones satisfy (II).

In order to prove that (II) implies (III), we let U^j_ℓ be the components of U_ℓ, $1 \leq \ell \leq L$. Now after a suitable translation apply a rotation (taking the axis of the cone C'_ℓ into the line $\{(0,\ldots,0,x_n)\colon x_n \in \mathbf{R}\}$) to the set U^j_ℓ. By taking possibly smaller sets U^j_ℓ and smaller ε, we may decompose these into cubes of side length at most $2\varepsilon/\sqrt{n}$, $U^j_\ell = \bigcup_k Q^j_\ell(k)$, where for fixed j and ℓ each of these cubes intersects at most 3^n-1 of the others and still satisfies iii). In this case $\partial\Omega \cap Q^j_\ell(k)$ is a restriction of the graph of a Lip 1 function of the first n-1 variables which has Lipschitz norm bounded by $\cot\theta$ (θ=vertex angle of C'). Since each $x \in \mathbf{R}^n$ can belong to at most one component of U_ℓ, then $\sum \chi_{U^j_\ell}(x) \leq 3^n \sum \chi_{U^j_\ell} \leq 3^n L$ and Ω satisfies all the conditions of (III).

Next we show that (III) implies (IV).[a)] First we adjust our open sets $\{U_j\}$ by taking instead U'_j consisting of the union of all cubes of side

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a) A simpler proof of (III) implies (IV) now appears in [5].

length $2\varepsilon/\sqrt{n}$ which are contained in U_j, then it is clear that $\{U_j'\}$ satisfies all the properties of (III). We show that since each point belongs to at most N of the sets U_j, then any cube Q with side length $2\varepsilon/\sqrt{n}$ intersects at most $N(2^n+1)$ of the sets U_j'. By the definition of U_j', if $U_j' \cap Q \neq \phi$, then there is a cube Q_j of side length $2\varepsilon/\sqrt{n}$ contained in U_j with $Q_j \cap Q \neq \phi$. By III iii) each vertex of Q (of which there are 2^n) belongs to at most N such Q_j. If Q_k is any other such cube which intersects Q nontrivially, then it must in fact coincide with Q since it can't contain a vertex, but there can be at most N such occurrences. This verifies that Q intersects at most $N(2^n+1)$ of the U_j'.

The final implication (IV) $\rightarrow$ (I) is obvious. □

In view of the theorem, we shall, in the sequel, say that $\partial\Omega$ is *minimally smooth* if any one of properties (I)-(IV) is satisfied. Now we would like to make a final comparison between the conditions on Ω here and those in Adams [1, p. 66]. The *strong local Lipschitz* property (4.5) in [1] is the same as $\partial\Omega$ being minimally smooth. The *uniform cone property* (4.4) of [1] is slightly weaker than our conditions in that condition (i) of property (I) is replaced by

$$\{x \in \Omega: \ \operatorname{dist}(x,\partial\Omega) < \varepsilon\} \subset \bigcup_j U_j \ .$$

Our property in effect guarantees a uniform overlap of the U_j's so that the corresponding partition of unity has uniform C^r norm. For completeness we give an example of a bounded $\Omega \subset \mathbf{R}^2$ with the uniform cone property whose boundary is not minimally smooth. Let $\Omega_0 := (-\frac{1}{2},1)\times(-\frac{1}{2},0)$ and define $R_j := (\frac{3}{4}\,2^{-j},2^{-j})\times[0,\frac{1}{3}\,2^{-j-2})$ $j = 0,1,2,\ldots$, then

$$\Omega := \Omega_0 \cup (\bigcup_j R_j)$$

is a bounded, open, connected subset of $\mathbf{R}^2$ with the uniform cone property. Indeed, it is possible to find a collection $\{U_1,\ldots,U_4\}$ and corresponding cones $\{C_1,\ldots,C_4\}$ with vertex angle $\pi/8$ and height $1/12$ to satisfy all the required conditions of the uniform cone property, but $\partial\Omega$ is not minimally smooth.

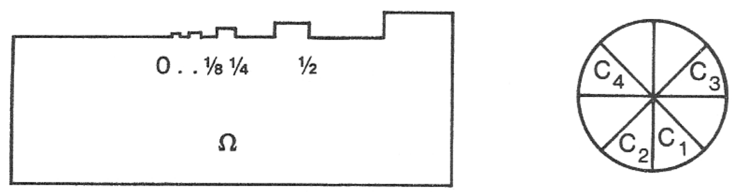

Figure 1.

For an open set Ω and r a nonnegative integer, the Sobolev seminorm is defined by

$$|f|_{p,r} := \sum_{|\nu|=r} \|D^{\nu} f\|_{L_p(\Omega)}$$

where $D^{\nu} f$ is the ν-th distributional derivative of f. We set the Sobolev space $W_p^r(\Omega)$ to consist of all f so that $|f|_{p,r}$ is finite. The K functional for $L_p(\Omega)$ and $W_p^r(\Omega)$ is defined as

$$K(f,t;\ L_p(\Omega),\ W_p^r(\Omega)) = \inf_{f=g+h} \{\|g\|_p + t|h|_{p,r}\} .$$

In [6] Johnen and Scherer showed that this K functional is connected to the r-th modulus of continuity in $L_p(\Omega)$,

$$\omega_r(f,t)_p := \sup_{|h|\leq t} \|\Delta_h^r f\|_{L_p(\Omega(rh))}$$

where $\Delta_h^r f$ is the r-th difference of f with step size h, and $\Omega(rh) := \{x\colon x+sh \in \Omega,\ 0 \leq s \leq r\}$. We sketch a proof of this result which is somewhat different than the original in [6].

PROPOSITION 2. Suppose Ω is an open set with minimally smooth boundary and $1 \leq p < \infty$, then there exist constants $c_1, c_2 > 0$ such that for each $f \in L_p(\Omega)$,

$$(1)\quad c_1\omega_r(f,t)_p \leq K(f,t^r;\ L_p(\Omega),\ W_p^r(\Omega)) \leq c_2\omega_r(f,t)_p,\quad 0 \leq t \leq 1.$$

PROOF. As it was pointed out in Lemma 1 of [6], the left hand inequality follows for any open set. Indeed, if $g \in W_p^r(\Omega)$, then

$$\|\Delta_h^r f\|_{L_p(\Omega(rh))} \leq 2^r \|f-g\|_p + \|\Delta_h^r g\|_{L_p(\Omega(rh))} \leq cK(f,t^r; L_p(\Omega), W_p^r(\Omega))$$

where the last inequality follows since

$$\Delta_h^r g(x) = \int_0^r \sum_{|\nu|=r} D^\nu g(x+\zeta h)h^\nu M_r(\zeta)d\zeta$$

and M_r is the B spline of order r with knots $0,1,\ldots,r$.

For the right hand inequality in (1), we first set $U_0 := \Omega^{\varepsilon/2}$ ($C_0 := C_1$ will do) and denote by $\{\phi_\ell\}_{\ell=0}^L$ the partition of unity, subordinate to $\{U_\ell\}_{\ell=0}^L$, as constructed in [8, p. 190] which has the properties: the ϕ_ℓ are nonnegative and supported in U_ℓ, $\sum \phi_\ell \equiv 1$ on Ω, and $\|D^\nu \phi_\ell\|_\infty$ are uniformly bounded for $|\nu| \leq r$ and $0 \leq \ell \leq L$.

Since $\omega_r(f)$ is nondecreasing and $\omega_r(f,ct) \leq (1+c)^r \omega_r(f,t)$ it suffices to prove the inequality for $0 < t \leq t_0$ where t_0 is fixed (depending at most on r and Ω). We set $t_0 := (6r^3\sqrt{n})^{-1}\ \varepsilon \sin\theta$ (θ the common vertex angle of the cones) to guarantee that we are always working within Ω. For fixed $0 < t \leq t_0$ and $0 \leq \ell \leq L$, define

$$(2)\quad g_\ell(x) := \int_{U^r} [I - (-1)^r \Delta^r_{t(h_\ell+u_1+\ldots+u_r)}] f(x) du_1 \ldots du_r, \quad x \in U_\ell \cap \Omega$$

where h_ℓ is the vector of length $3r^2\sqrt{n}(\sin\theta)^{-1}$ whose direction is parallel to the axis of C_ℓ (direction e_1 will do for $\ell = 0$) and U^r is the r-fold product of the unit cube $U = \{(y_1,\ldots,y_n): 0 \leq y_i \leq 1\}$ in R^n. Notice that for each $x \in U_\ell \cap \Omega$, the integration in (2) is performed within $x + C_\ell \subset \Omega$ and so g_ℓ is well defined. On Ω, set

$$g := \sum_{\ell=0}^L \phi_\ell g_\ell,$$

then the proof will be completed once $\|f-g\|_p$ and $t^r\|D^\nu g\|_p$, $|\nu| = r$, are shown to be no larger than $c_2\omega_r(f,t)_p$. The estimate for $f-g$ follows straight away,

$$\begin{aligned}(3)\quad \|f-g\|_{L_p(\Omega)} &= \|\sum_{\ell=0}^L \phi_\ell \int_{U^r} [\Delta^r_{t(h_\ell+u_1+\ldots+u_r)} f] du_1 \ldots du_r\|_{L_p(\Omega)} \\ &\leq (L+1)\omega_r(f,\gamma t) \leq c_2\omega_r(f,t)\end{aligned}$$

where $\gamma := 4r^2\sqrt{n}(\sin\theta)^{-1}$. For the terms $t^r\|D^\nu g\|_p$ we use Leibnitz' rule for products on $U_\ell \cap \Omega$ applied to

$$g = \sum_{j=0}^{L} \phi_j g_j = g_\ell + \sum_{\substack{j=0\\ j\neq\ell}}^{L} (g_j - g_\ell)\phi_j .$$

The interpolation inequality (see [1], p. 75) together with the identity $(g_j-g_\ell) = (f-g_\ell) - (f-g_j)$ on $U_\ell \cap U_j \cap \Omega$ and estimates of the type in (3), show that it is enough to establish estimates

$$(4) \qquad \|D^\nu \tilde{g}\|_{L_p(U_\ell\cap\Omega)} \leq ct^{-r}\omega_r(f,t)_{L_p(\Omega)}$$

where by (2) $\tilde{g}$ is a typical term of g_ℓ,

$$\tilde{g}(x) := (-1)^{k+1}\binom{r}{k}\int_{U^r} f(x+kt[h_\ell+u_1+\ldots+u_r])du_1\ldots du_r,$$

for $1 \leq k \leq r$. If $|\nu| = r$, then

$$|D^\nu\tilde{g}(x)| = \binom{r}{k}(kt)^{-r}\left|\int \prod_{i=1}^{n} \Delta_{kte_i}^{\nu_i} f(x+kt[h_\ell+\sum \sigma_{ij}]) \prod_{i,j} d\sigma_{ij}\right|$$

where the sum and integral run over the indices $(1\leq i\leq n, \nu_i<j\leq r)$ since r of the σ_{ij}'s have integrated D^ν to give the differences. Next we write the mixed differences in terms of pure differences (see e.g. [6], Lemma 2)

$$\prod_{i=1}^{n} \Delta_{kte_i}^{\nu_i} f(y) = \sum_{D\in D^*}(-1)^{|D|}\Delta_{h(D)}^{r} f(y+b(D))$$

where D* is all subsets of $\{1,2,\ldots,r\}$, $h(D) := -\sum_{j\in D}\frac{kt}{j}\tilde{e}_j$, $b(D) := \sum_{j\in D} kt\tilde{e}_j$,

and the $\tilde{e}_j$'s are just the unit coordinate vectors e_i repeated according to their multiplicity ν_i. Substituting this into the previous inequality and integrating the p-th power over $U_\ell \cap \Omega$, we see that the argument of f stays within Ω by our selection of t_0 and so inequality (4) follows as desired.□

In [6] and [4] inequalities between moduli of continuity were established for domains with minimally smooth boundaries. We indicate how these results follow from Proposition 2 and the extension theorem for Sobolev spaces. A simple illustration is the Marchaud inequality

$$\omega_1(f,t)_{p,\Omega} \leq ct[\|f\|_p + \int_t^1 \frac{\omega_2(f,s)_{p,\Omega}}{s}\frac{ds}{s}], \quad 0 \leq t \leq 1 \tag{5}$$

which follows immediately for $\Omega = \mathbf{R}^n$ by applying L_p norms to the well known identity (choosing N so that $1/2 < 2^N t \leq 1$)

$$\Delta_h g = -\sum_{j=0}^{N-1} 2^{-j-1}\Delta^2_{2^j h} g + 2^{-N}\Delta_{2^N h} g$$

and using the fact that $\omega_2(g,s)_{p,\mathbf{R}^n}$ is monotone to get

$$2^{-j-1}\omega_2(g,2^j t) \leq (\ln 2)^{-1} t \int_{2^j t}^{2^{j+1} t} \omega_2(g,s)s^{-2}ds.$$

For Ω with $\partial\Omega$ minimally smooth, we denote by E the extension operator [8, p. 180] which maps $W^j_p(\Omega)$ boundedly to $W^j_p(\mathbf{R}^n)$, $0 \leq j \leq r$, such that Ef coincides a.e. with f on Ω. By Proposition 2 we clearly have that

$$\omega_r(f,t)_{p,\Omega} \leq \omega_r(Ef,t)_{p,\mathbf{R}^n} \leq c\omega_r(f,t)_{p,\Omega}, \quad 0 \leq t \leq 1 \tag{6}$$

since $K(Ef, t^r; L_p(\mathbf{R}^n), W^r_p(\mathbf{R}^n)) \leq cK(f,t^r; L_p(\Omega), W^r_p(\Omega))$, and so (5) follows from the corresponding result applied to $g = Ef$ on $\mathbf{R}^n$. In the remainder of the paper we exploit this pattern and also give a simplification of an inequality on $\mathbf{R}^n$ which is used to embed Besov spaces. Of course, quite a bit of work is hidden in the construction of the extension operator, but recently [5, p. 90] and [7] give straightforward constructions which go back to the original ideas of Whitney. These constructions produce extension operators $E = E_r$, depending on r, which map $W^j_p(\Omega)$ to $W^j_p(\mathbf{R}^n)$, $0 \leq j \leq r$, which is all we require.

On $\mathbf{R}^n$, using the Sobolev embedding theorem it is not too difficult to prove (see [4]) that for $r \geq n$

$$g^{**}(t) \leq c \begin{cases} \int_{t^{1/r}}^1 s^{-n/p-1}\omega_r(g,s)_p \; ds, & 0 < t \leq 1 \\ t^{-1/p}\|g\|_p, & t \geq 1 \end{cases} \tag{7}$$

where the case $t \geq 1$ is just Hölder's inequality. Here $g^{**}(t) = t^{-1}\int_0^t g^*$ where g^* is the decreasing rearrangement of g. We outline the proof presented in [4] for $\Omega = \mathbf{R}^n$ and $0 < t \leq 1$. Fix t and choose g_k (from

Proposition 2 and the definition of the K functional) $k = 1,2,\ldots$, which satisfies

$$(8) \qquad \|g-g_k\|_p + 2^k t^r |g_k|_{p,r} \leq c\omega_r(g,2^{k/r}t)_p$$

and set $\psi_k := g_k - g_{k+1}$, then writing

$$(9) \qquad g = g-g_1 + \sum_{k=1}^{N-1} \psi_k + g_N$$

and estimating, we see that

$$(10) \qquad g^{**}(t^r) \leq (g-g_1)^{**}(t^r) + \sum_{k=1}^{N-1} \|\psi_k\|_\infty + \|g_N\|_\infty .$$

By Hölder's inequality

$$(11) \qquad (g-g_1)^{**}(t^r) \leq t^{-r/p}\|g-g_1\|_p \leq ct^{-r/p}\omega_r(g,t)_p,$$

while

$$(12) \qquad \|\psi_k\|_\infty \leq cs^{-n/p}[\|\psi_k\|_p + s^r|\psi_k|_{p,r}]$$

by applying the Sobolev embedding theorem ($W_p^r \subset L^\infty$, $r \geq n$) to the function $h_s(x) = h(sx)$. But (12) and (8) show by selecting $s = 2^{k/r}t$ that

$$(13) \quad \|\psi_k\|_\infty \leq c(2^{k/r}t)^{-n/p}\omega_r(g,2^{k/r}t)_p \leq c \int_{2^{k/r}t}^{2^{(k+1)/r}t} s^{-n/p}\omega_r(g,s)_p \frac{ds}{s} .$$

Using again the Sobolev estimate together with (8) we have

$$(14) \quad \|g_N\|_\infty \leq c(\|g_N\|_p + |g_N|_{p,r}) \leq c[\|g\|_p + \|g-g_N\|_p + |g_N|_{p,r}]$$
$$\leq c[\|g\|_p + \omega_r(g,1)_p] \leq c\|g\|_p ,$$

since we can choose N so that $2^{N/r}t \leq 1 \leq 2^{(N+1)/r}t$. Now summing over k and using (10), (13), and (14), inequality (7) follows. For Ω, we use the result on $\mathbb{R}^n$ together with (6). As was the point in [4] once the inequality (7) is established, then the Besov space embeddings into Lorentz spaces

(15) $$B_p^{\theta,a}(\Omega) \to L_{q,a}(\Omega), \quad \theta = \frac{n}{p} - \frac{n}{q}$$

follow by applying the Lorentz norm to both sides of (7) and using Hardy's inequality for integral averages. Recall here that

$$\|f\|_{B_p^{\theta,a}} := \{\int_0^1 [t^{-\theta}\omega_r(f,t)_p]^a \frac{dt}{t}\}^{1/a} + \|f\|_p$$

where $r = [\theta] + 1$, and the Lorentz norm is given by

$$\|f\|_{q,a} := \{\int_0^\infty [f^*(t)t^{1/q}]^a \frac{dt}{t}\}^{1/a} .$$

Using this embedding, we can give an easy proof of another Besov space embedding on $\mathbf{R}^n$ (see [4], p. 74-75), namely

(16) $$\omega_r(g,t)_q \le c\int_0^t s^{-\theta}\omega_r(g,s)_p \frac{ds}{s}$$

where $1 \le p \le q < \infty$, $\theta = \frac{n}{p} - \frac{n}{q}$, and $r \ge n$. Then, as before, if we apply this result to $g = Ef$, by (6) the same inequality will follow for Ω with minimally smooth boundary. To prove (16) we apply (15) to $\Delta_h^r g$ and use the well known fact (following from Hölder's inequality) that $B_p^{\theta,1} \to B_p^{\theta,q}$, in order to obtain for $|h| \le t$

$$\|\Delta_h^r g\|_q \le c[\|\Delta_h^r g\|_p + \int_0^1 s^{-\theta}\omega_r(\Delta_h^r g,s)_p \frac{ds}{s}]$$

$$\le c[\omega_r(g,t)_p + \int_0^t s^{-\theta}\omega_r(g,s)_p \frac{ds}{s} + \int_t^1 s^{-\theta-1}ds\ \omega_r(g,t)_p]$$

$$\le c \int_0^t s^{-\theta}\omega_r(g,s)_p \frac{ds}{s} .$$

The second inequality follows from

$$\omega_r(\Delta_h^r g,s) \le 2^r \min\{\omega_r(g,s),\omega_r(g,t)\}$$

while the third follows since

$$t^{-\theta}\omega_r(g,t) \le 2^r t^{-\theta}\omega_r(g,t/2) \le 2^r(\ell n\ 2)^{-1}\int_{t/2}^t s^{-\theta}\omega_r(g,s) \frac{ds}{s} .$$

Again, as in [4], we may apply a Lorentz norm to (16) to obtain the Besov embedding

$$B_p^{\theta+\lambda,a}(\Omega) \to B_q^{\lambda,a}(\Omega)$$

where $\lambda > 0$, $1 \leq a \leq \infty$, $\theta = \frac{n}{p} - \frac{n}{q}$, and $1 \leq p \leq q < \infty$.

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Department of Mathematics and Statistics
University of South Carolina
Columbia, S.C., U.S.A. 29208