ON AN INEQUALITY FOR THE SHARP FUNCTION

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The purpose of this note is to refine a rearrangement inequality for $f^{\#}$ in terms of the maximal rearrangement f^{**} for locally integrable functions on $\mathbb{R}^n$. One consequence of this inequality is an improvement and clarification of the proof that interpolation between L^1 and BMO "coincides" with that between L^1 and L^∞.

The "oscillation" of a locally integrable function f on $\mathbb{R}^n$ is gauged by its sharp function (cf [1])

$$(1)\quad f^{\#}(x) = \sup_{Q \ni x} \left\{ \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy \right\}, \qquad x \in \mathbb{R}^n .$$

Here f_Q denotes the average $|Q|^{-1}\int_Q f(y)\,dy$ of f over a cube Q with sides parallel to the coordinate axes, and the supremum in (1) is taken over all such cubes Q containing the point x. Observe that the sharp function ignores constants: if f and g differ by a constant, then $f^{\#}$ and $g^{\#}$ coincide.

The decreasing rearrangement of f will be denoted by f^*, and its average $t^{-1}\int_0^t f^*(s)\,ds$ by $f^{**}(t)$. The latter function is "equivalent" to the decreasing rearrangement of the Hardy-Littlewood maximal function Mf of f in the sense that the ratio of f^{**} and $(Mf)^*$ is contained between positive constants independent of f (cf [1, Theorem 1.3]).

It follows immediately from (1) that $f^{\#} \le 2Mf$ and hence, by the preceding remarks, that $f^{\#*} \le cf^{**}$. There is also a result in the opposite direction, that is, it is possible to estimate f^{**} in terms of $f^{\#*}$. In fact, it was shown in [1, Corollary 4.2] that the inequality

$$(2)\quad f^{**}(t) \le c\int_t^\infty f^{\#*}(s)\frac{ds}{s} + f^{**}(+\infty), \qquad 0 < t < \infty ,$$

holds.

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The importance of this result was demonstrated in [1] where a "local" version of (2) easily produced the well-known John-Nirenberg lemma. Nevertheless, the presence of the term $f^{**}(+\infty)$ on the right of (2) is bothersome in several of the applications in [1], and this led us to seek a more convenient formulation. In view of a remark made earlier, the first term on the right of (2) is unchanged when a constant γ is subtracted from f. In this note it will be shown that the finiteness of this integral guarantees the existence of a unique constant γ such that $(f-\gamma)^{**}(+\infty) = 0$, and hence, if f is replaced by $f-\gamma$ in (2), the troublesome term at infinity does not arise. The main result is thus as follows.

THEOREM 1. *Let* f *be locally integrable on* $\mathbb{R}^n$ *and suppose*

$$(3) \quad \int_1^\infty f^{\#*}(s)\,\frac{ds}{s} < \infty .$$

Then there is a unique constant γ $(= \lim_{|Q|\to\infty} f_Q)$ *such that*

$$(4) \quad (f-\gamma)^{**}(t) \le c\int_t^\infty f^{\#*}(s)\frac{ds}{s}, \qquad 0 < t < \infty .$$

The proof requires a pair of lemmas.

LEMMA 2. *If* $Q_0 \subset Q_1 \subset \mathbb{R}^n$, *then*

$$(5) \quad |f_{Q_0} - f_{Q_1}| \le c\int_{\frac{1}{2}|Q_0|}^\infty f^{\#*}(s)\frac{ds}{s},$$

where c *depends only on the dimension* n.

Proof. From inequality (4.23) of [1] it follows that

$$(6) \quad [(f-f_Q)\chi_Q]^{**}(t) \le c\int_t^{|Q|} f^{\#*}(s)\frac{ds}{s}, \qquad 0 < t < \frac{|Q|}{2},$$

for any cube Q. But if $0 \le t \le |Q|$, then $[\lambda\chi_Q]^{**}(t) = \lambda$, so

$$|f_{Q_0} - f_{Q_1}| = [(f_{Q_0}-f_{Q_1})\chi_{Q_0}]^{**}\left(\frac{|Q_0|}{2}\right)$$
$$\le [(f-f_{Q_0})\chi_{Q_0}]^{**}\left(\frac{|Q_0|}{2}\right) + [(f-f_{Q_1})\chi_{Q_1}]^{**}\left(\frac{|Q_0|}{2}\right),$$

since $(g+h)^{**} \le g^{**} + h^{**}$ and $Q_0 \subset Q_1$. Together with (6) (applied for $Q = Q_0$ and $Q = Q_1$) this gives the desired inequality (5).

LEMMA 3. <u>If condition</u> (3) <u>holds, then</u> $\lim_{|Q| \to \infty} f_Q$ <u>exists</u>.

<u>Proof</u>. If $Q(k)$ denotes the cube with side length 2^k and centered at the origin, then it follows directly from (3) and (5) that $\{f_{Q(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence. Let γ be its limit. For each $\varepsilon > 0$ it is possible, by (3), to choose a corresponding $M > 0$ such that

$$(7) \quad c\int_{M/2}^{\infty} f^{\#*}(s)\frac{ds}{s} < \frac{\varepsilon}{3} ,$$

where c is the constant in (5). Then k may be chosen so large that $|f_{Q(k)} - \gamma| < \varepsilon/3$ and $|Q(k)| > M$. In that case, if Q is any cube with $|Q| > M$ and if Q' is any cube containing both $Q(k)$ and Q, then it follows from (5), (7), and the choice of k that

$$|f_Q - \gamma| \le |f_Q - f_{Q'}| + |f_{Q'} - f_{Q(k)}| + |f_{Q(k)} - \gamma| < \varepsilon .$$

<u>Proof of Theorem 1</u>. Let $\gamma = \lim_{|Q| \to \infty} f_Q$, which exists by virtue of the preceding lemma. Fix $t > 0$ and let $\varepsilon > 0$ be arbitrary. Then there is a cube Q with measure exceeding $2t$ and satisfying $|f_Q - \gamma| < \varepsilon$. Consequently, by (6),

$$\begin{aligned}[(f-\gamma)\chi_Q]^{**}(t) &\le [(f-f_Q)\chi_Q]^{**}(t) + |f_Q - \gamma| \\ &\le c\int_t^{\infty} f^{\#*}(s)\frac{ds}{s} + \varepsilon .\end{aligned}$$

If now $Q \uparrow \mathbb{R}^n$, the monotone convergence theorem shows that $[(f-\gamma)\chi_Q]^{**}(t) \uparrow (f-\gamma)^{**}(t)$. Hence the preceding estimate, after ε is allowed to decrease to zero, produces the desired inequality (4). For the uniqueness, note that if (4) holds for each of two constants γ_1 and γ_2, then for any $t > 0$,

$$|\gamma_1 - \gamma_2| = |\gamma_1 - \gamma_2|^{**}(t) \le (f-\gamma_1)^{**}(t) + (f-\gamma_2)^{**}(t)$$
$$\le 2c\int_t^\infty f^{\#*}(s)\frac{ds}{s} ,$$

and this, by (3), tends to 0 as $t \to \infty$.

Theorem 1 has a bearing on the identification of the interpolation spaces between the Lebesgue space L^1 and the space BMO of functions of bounded mean oscillation. The latter space consists of all equivalence classes (denoted by F) modulo constants of functions f for which $f^\#$ is bounded. Since the sharp function is invariant under addition of constants the notation $F^\# = f^\#$ for any representative f of F is meaningful. Hence, with the norm

(8) $$\|F\|_{BMO} = \|F^\#\|_{L^\infty} ,$$

BMO is a Banach space.

It was established in [1,§6] that the Peetre K-functional

$$K(f,t;L^1,BMO) = \inf_{f=g+h} (\|g\|_{L^1} + t\|H\|_{BMO})$$

is given by

(9) $$K(f,t;L^1,BMO) \sim tf^{\#*}(t) , \qquad 0 < t < \infty ,$$

for any f in $L^1 + BMO$.

Notice, however, that (L^1,BMO) is not, strictly speaking, a compatible Banach couple in that one space consists of functions and the other of equivalence classes of such functions modulo constants. The difficulty is resolved by introducing the space $\mathcal{L}^1$ consisting of all equivalence classes F for which the norm $\|F\|_{\mathcal{L}^1} = \inf_{f\in F} \|f\|_{\mathcal{L}^1}$ is finite. Of course there will be precisely one representative f in L^1 of each equivalence class F in $\mathcal{L}^1$. With this, it is clear from (9) that the analogous result

(10) $$K(F,t;\mathcal{L}^1,BMO) \sim tF^{\#*}(t)$$

holds for the Banach couple (L^1, BMO). Hence, with the aid of Theorem 1, the (θ,q)-interpolation spaces may be identified as follows.

COROLLARY 4. *Suppose* $0 < \theta < 1$, $0 < q \le \infty$, *and let* $1/p = 1-\theta$. *Then, for any* F *in* $L^1 + \mathrm{BMO}$,

$$\|F\|_{(L^1,\mathrm{BMO})_{\theta,q}} \sim \|F^\#\|_{L^{pq}} = \|f^\#\|_{L^{pq}} \sim \|f-\gamma\|_{L^{pq}},$$

where f *is any representative of* F *and* $\gamma = \lim_{|Q|\to\infty} f_Q$. *Hence*

$$(L^1,\mathrm{BMO})_{\theta,q} = (L^1,L^\infty)_{\theta,q} = L^{pq},$$

in the sense that if F *belongs to* $(L^1,\mathrm{BMO})_{\theta,q}$, *then there is a unique representative* $g = f-\gamma$ *of* F *such that*

$$\|g\|_{L^{pq}} \le c\|F\|_{(L^1,\mathrm{BMO})_{\theta,q}},$$

and, conversely, that if f *belongs to* L^{pq}, *then*

$$\|F\|_{(L^1,\mathrm{BMO})_{\theta,q}} \le c\|f\|_{L^{pq}},$$

where F *is the coset containing* f.

REMARKS. (i) The identification of the interpolation spaces, as in Corollary 4, does not require the complete knowledge (9) of the K-functional (cf [1, Corollary 4.4]).

(ii) The Hilbert transform and its n-dimensional analogues (the Riesz transforms) are well defined on L^p $(1 \le p < \infty)$ as the principal value integrals

$$(11) \quad R_j f(x) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} f(y) K_\varepsilon(x-y)\,dy$$

where $K_\varepsilon(y) = K(y)\chi_{(\varepsilon,\infty)}(|y|)$ and $K(y) = c_n y_j/|y|^{n+1}$. On L^∞, however, these integrals will normally not exist and so must be modified [2] according to

$$(12)\quad \tilde{R}_j f(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} f(y)[K_\varepsilon(x-y) - K_1(-y)]dy\ .$$

To use the facts that the "Hilbert transform" maps L^∞ into BMO and say, H^1 into L^1 (or, L^1 into weak L^1) together with interpolation in order to obtain intermediate results, the ambiguity between the definitions (11) and (12) must be resolved. The inequality one is able to obtain [1, Corollary 5.6] is

$$(13)\quad (R_j f)^*(t) \le c\left\{\frac{1}{t}\int_0^t f^*(s)ds + \int_t^\infty f^*(s)\frac{ds}{s}\right\}$$

where the left hand side makes sense whenever the right hand side of the inequality is finite. This inequality is derived by using the finiteness of $\int_t^\infty f^*(t)\frac{dt}{t}$ to establish that $\int_{R^n} |f(y)|\ |K_1(-y)|dy < \infty$ and so (11) and (12) differ by at most a constant γ which is controlled according to Theorem 1.

REFERENCES

1. Bennett, C. and R. Sharpley, Weak-type inequalities for H^p and BMO, Proc. Sympos. Pure Math. 35I (1979), 201-229.

2. Fefferman, C. and E. M. Stein, H^p spaces of several variables, Acta Math. 129 (1972), 137-193.

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