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Multilinear weak type interpolation of m n-tuples with applications

by

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Abstract. Two versions of a multilinear weak type interpolation theorem for rearrangement invariant spaces $\Lambda_a(X)$ are obtained by extending the Calderón operator S from the linear case. Specific applications of each version of the theorem are given to certain bilinear operators which include integral operators, tensor products, convolution, and product operators.

§ 1. Introduction. This paper is a natural extension of the methods and ideas developed in [3] and [13] to study weak type linear or sublinear operations. The setting here is multilinear operations satisfying m initial estimates. Using an appropriate modified Calderón operator and Hölder's inequality, we are able to obtain a multilinear (multisublinear) version of the Stein–Weiss interpolation theorem for rearrangement invariant spaces (Theorem (3.4)) and then apply the result to integral operators (Corollary (4.3)) and tensor products (Corollary (4.6)). Using the same techniques with slightly different estimates, we deduce convolution and product operator theorems (Corollaries (4.7) and (4.8)).

In his extensive work examining the above operators on Lorentz $L^{p,q}$ ([9], [11]) and Orlicz spaces ([10], [11]), O'Neil systematically attacked each problem in the following manner.

Utilizing the endpoint estimates, he would derive for each type of operator a "basic inequality". Then, using this in conjunction with a "fundamental condition" relating the intermediate spaces, he was able in most cases to give necessary and sufficient conditions for the operators to be bounded. As a guide we outline the procedures followed for convolution operators. Convolution operators are defined as bilinear operators C which satisfy the conditions $\|C(f,g)\|_1 \leqslant \|f\|_1\|g\|_1$, $\|C(f,g)\|_\infty \leqslant \|f\|_1\|g\|_\infty$, and $\|C(f,g)\|_\infty \leqslant \|f\|_\infty\|g\|_1$, where $\|h\|_p$ is the Lebesgue p-norm of h. From these initial inequalities O'Neil [9] derived the "basic inequality for convolution"

$$C(f,g)^{**}(t) \leqslant tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)\,ds \tag{1.1}$$

and then proceeded to show that C satisfied

(1.2) $$\|C(f,g)\|_{r,1/\gamma} \leqslant \text{const}\ \|f\|_{p,1/\alpha}\|g\|_{q,1/\beta}$$

so long as the indices of the intermediate spaces satisfied the "fundamental condition for convolution"

(1.3) $$t\cdot t^{1/r} \leqslant \text{const}\ t^{1/p}t^{1/q}, \quad \text{for all } t$$

and the secondary index condition for convolution

(1.4) $$\gamma \leqslant \alpha+\beta,$$

where $\|h\|_{p,q}$ denotes the Lorentz $L^{p,q}$ norm of h.

In this paper we show that various operator estimates of which relation (1.1) is a special case can be established in a unified manner using Calderón's weak type theory. Inequalities of the form (1.1) are obtained automatically for each class of operators by evaluating Calderón's maximal operator S for the interpolation scheme under consideration and by using the fact that the relation

(1.5) $$T(f,g)^{**} \leqslant S(f^*,g^*) \quad \text{a.e.}$$

holds. The appropriate "fundamental condition" for the class is obtained by observing minimal conditions the operators must satisfy in order to be bounded. In short, the main purpose of this paper is to exhibit an easy way to determine the form of the "basic inequalities" for various classes of operators, establish necessary conditions, and place the existing theorems in an interpolation theoretic framework.

§ 2. $\Lambda_\alpha(X)$ spaces and Calderón's operator. The spaces $\Lambda_\alpha(X)$ [13] are generalizations of the Lorentz $L^{p,q}$ spaces ([5], [3]) which retain many of their properties. We mention only those properties which are necessary for the development of the multilinear theory. For the proofs of these facts we refer the reader to [13].

The *distribution function* of a measurable function f is defined by $m_{|f|}(t) = m\{s \mid |f(s)|>t\}$, where m is the measure involved. The *decreasing rearrangement* f^* of a measurable function is the non-increasing, right continuous inverse on $(0,\infty)$ of $m_{|f|}(t)$. Two measurable functions are called *equimeasurable* if they have the same distributions. The *averaged rearrangement* of a function f is defined by $f^{**}(t) = 1/t\int_0^t f^*(s)\,ds$.

A *rearrangement invariant Banach function space* is a Banach space of measurable functions which satisfy the following properties:

(i) $|g|\leqslant|f|$ a.e., $f\in X$ implies $g\in X$ and $\|g\|\leqslant\|f\|$,

(ii) $0\leqslant f_n\nearrow f$ a.e., $f_n\in X$, and $\|f_n\|\leqslant M$, then $f\in X$ and $\|f\|\leqslant M$,

(iii) $mE<\infty$ implies there exists $C_E>0$ so that $\int_E f\,ds \leqslant C_E\|f\|$ independent of f,

(iv) $mE<\infty$ implies $\|\chi_E\|<\infty$, and

(v) f and g equimeasurable, $f\in X$ implies $g\in X$ and $\|f\|=\|g\|$.

The fundamental function of a rearrangement invariant space X, $\varphi_X(t) = \|\chi_E\|_X$, $mE=t$, can be shown to be concave and nondecreasing for $t>0$. The *associate space* X' of a rearrangement invariant space is the space of functions g such that $\|g\|_{X'} = \sup_{\|f\|_X\leqslant 1}\int f^*g^*\,ds$ is finite. It is not difficult to show that $\varphi_X(t)\varphi_{X'}(t)=t$. Two classes $\mathscr{U}$ and $\mathscr{L}$ of these spaces are particularly important in applications. X belongs to $\mathscr{U}$ if for some $M>0$ and $\theta<1$,

$$\frac{\varphi_X(s)}{\varphi_X(t)} \leqslant \text{const}\ (s/t)^\theta \quad \text{when} \quad M<s/t,$$

while $X\in\mathscr{L}$ means there is a $\delta>0$ and $\theta>0$ so that

$$\frac{\varphi_X(s)}{\varphi_X(t)} \leqslant \text{const}\ (s/t)^\theta \quad \text{when} \quad s/t<\delta.$$

We define $\Lambda_\alpha(X)$, $0<\alpha\leqslant 1$, as the space of all measurable functions such that the norm

(2.1) $$\|f\|_{\Lambda_\alpha}\|f\|_{\Lambda_\alpha(X)} = \left\{\int_0^\infty \big(f^{**}(s)\varphi_X(s)\big)^{1/\alpha}\frac{ds}{s}\right\}^\alpha$$

is finite. If X belongs to $\mathscr{U}\cap\mathscr{L}$, it is not hard to see that $\Lambda_\alpha(X)$ is a rearrangement invariant space with a fundamental function equivalent to φ_X. We define

$$\Lambda(X) = \left\{f \mid \|f\|_{\Lambda(X)} = \int f^*\,d\varphi_X < \infty\right\}$$

and

$$M(X) = \{f \mid \|f\|_{M(X)} = \sup_t \{f^{**}(t)\varphi_X(t)\} < \infty\}.$$

It is not hard to show [12] that

(2.2) $$\Lambda(X) \subseteq X \subseteq M(X)$$

with continuous embeddings and that $\Lambda(X)$ and $M(X)$ have fundamental function φ_X. If $0\leqslant\alpha\leqslant\beta\leqslant 1$, then $\Lambda_\beta(X)\subseteq\Lambda_\alpha(X)$ with continuous embeddings where $\Lambda_0(X)\equiv M(X)$. If X belongs to $\mathscr{U}$, then we can show that $\Lambda_1(X)=\Lambda(X)$ with equivalent norms. If X belongs to $\mathscr{U}\cap\mathscr{L}$, then the simple functions with compact support form a dense subset of $\Lambda_\alpha(X)$, $0<\alpha\leqslant 1$. If the operators P and Q are defined by

(2.3) $$P(f)(t) = 1/t\int_0^t f(s)\,ds, \qquad Q(f)(t) = \int_t^\infty f(s)\frac{ds}{s},$$

then it can be shown that P and Q are bounded operators on $\Lambda_\alpha(X)$, $0\leqslant\alpha\leqslant 1$, when $X\in\mathscr{U}\cap\mathscr{L}$ ([2], [13]). Using these operators, one can also show [13] that

$$\varphi_X(t) \sim \int_0^t \varphi_X(s)\frac{ds}{s} \tag{2.4}$$

and

$$\frac{1}{\varphi_X(t)} \sim \int_t^\infty \frac{1}{\varphi_X(s)}\frac{ds}{s} \tag{2.5}$$

when $X \in \mathcal{U} \cap \mathcal{L}$.

We assume that the operators are bilinear in order to simplify the notation even though the proofs are valid for multi-sublinear operators. We do state the Stein–Weiss theorem, however, in its full form. A *weak type* $([X, Y], Z)$ *bilinear operator* is one such that

$$\|T(\chi_E, \chi_F)\|_{M(Z)} \leqslant \text{const}\, \varphi_X(mE)\varphi_Y(mF), \tag{2.6}$$

where the constant is independent of the sets E and F. We consider bilinear operators T which are of weak types $([X(j), Y(j)], Z(j))$ for $j = 1, \ldots, m$. For this interpolation scheme the function

$$\Psi(r, s; t) = \min_{1\leqslant j\leqslant m}\left(\frac{\varphi_{X(j)}(r)\varphi_{Y(j)}(s)}{\varphi_{Z(j)}(t)}\right) \tag{2.7}$$

is used to define a modified Calderón operator

$$S(f, g)(t) = \int_0^\infty\int_0^\infty f(r)g(s)\Psi(r, s; t)\frac{dr}{r}\frac{ds}{s} \tag{2.8}$$

which is maximal in the following sense:

THEOREM 2.1. *If T is a bilinear operator of weak types $([X(j), Y(j)], Z(j))$ with norm C_j, $1 \leqslant j \leqslant m$, then*

$$T(f, g)^{**}(t) \leqslant \max_{1\leqslant j\leqslant m} C_j S(f^*, g^*)(t) \tag{2.9}$$

for all simple functions f and g with compact support.

Proof. Since T is of weak types $([X(j), Y(j)], Z(j))$,

$$T(\chi_E, \chi_F)^{**}(t) \leqslant \frac{\|T(\chi_E, \chi_F)\|_{M(Z(j))}}{\varphi_{Z(j)}(t)} \leqslant C_j\frac{\varphi_{X(j)}(mE)\varphi_{Y(j)}(mF)}{\varphi_{Z(j)}(t)}, \quad 1 \leqslant j \leqslant m.$$

Therefore

$$T(\chi_E, \chi_F)^{**}(t) \leqslant \max_{1\leqslant j\leqslant m}(C_j)\Psi(mE, mF; t)$$

$$\leqslant \max_{1\leqslant j\leqslant m}(C_j)\int_0^\infty\int_0^\infty \chi_E^*(r)\chi_F^*(s)\Psi(r, s; t)\frac{dr}{r}\frac{ds}{s},$$

since $\chi_E^* = \chi_{(0,mE)}$, $\chi_F^* = \chi_{(0,mF)}$, and $\Psi(r, s; t)/(rs)$ is a non-increasing function both of r and s. Inequality (2.9) follows for simple f and g with compact support by applying the sublinearity of $T(\chi_E, \chi_F)^{**}$ in each variable to covariant decompositions of f and g.

Using this result, we are able to extend T in a unique manner:

THEOREM 2.2. *Suppose one of X or Y is separable. If T is a bilinear operator of weak types $([X(j), Y(j)], Z(j))$ of norm C_j, $j = 1, \ldots, m$, and*

$$\|S(f^*, g^*)\|_Z \leqslant c\|f\|_X\|g\|_Y \tag{2.10}$$

for all f in X and g in Y, then T has a unique extension to $X \times Y$ such that

$$\|T(f, g)\|_Z \leqslant c\,(\max_{1\leqslant j\leqslant m} C_j)\|f\|_X\|g\|_Y. \tag{2.11}$$

Proof. Suppose that X is separable. If we let E be a measurable set with finite measure, then by (2.10) $S(\chi_E^*, g^*)$ belongs to Z for each g in Y and must be finite almost everywhere. By slightly modifying the proof of Corollary 4.4 and Theorem 4.5 of [13], T has a unique extension to the pair (χ_E, g) such that

$$T(\chi_E, g)^{**}(t) \leqslant CS(\chi_E^*, g^*)(t)$$

where $C = \max_{1\leqslant j\leqslant m} C_j$. Taking an arbitrary simple function f with compact support, we may write f in covariant form

$$f(s) = \sum_{i=1}^k a_i \operatorname{sgn} f(s)\chi_{E_i}(s),$$

where $a_i \geqslant 0$ and $E_i \supseteq E_{i+1}$. Since the averaged rearrangement is sublinear, we have

$$T(f, g)^{**}(t) \leqslant \sum_{i=1}^k a_i T(\chi_{E_i}, g)^{**}(t) \leqslant C\sum_{i=1}^k a_i S(\chi_{E_i}^*, g^*)(t) = CS(f^*, g^*)(t)$$

by noticing $f^* = \sum_{i=1}^k a_i\chi_{E_i}^*$. But this together with inequality (2.10) shows that (2.11) holds for all g in Y and simple functions f with compact support. But the simple functions with compact support are dense in X, so $T(\cdot, g)$ has a unique extension to X for each g in Y so that (2.11) holds for all f in X. It is not hard to see that the extension remains bilinear.

Remark 2.3. If in Theorem 2.2 we can assume both X and Y are separable, then we can relax the definition of weak type $([X(j), Y(j)], Z(j))$ to

$$\sup_t\big(T(f, g)^*(t)\varphi_{Z(j)}(t)\big) \leqslant C_j\|f\|_{\Lambda(X(j))}\|g\|_{\Lambda(Y(j))}$$

for all simple functions f and g with compact support.

Remark 2.4. If all the endpoint spaces $X(j)$, $Y(j)$, and $Z(j)$ belong to $\mathcal{U} \cap \mathcal{L}$, then it is possible to show that S is a bilinear operator of weak types $([X(j), Y(j)], Z(j))$, $j = 1, \ldots, m$. Therefore in the "non-extreme" cases of weak interpolation, a necessary and sufficient condition for each bilinear operator of weak types $([X(j), Y(j)], Z(j))$ $1 \leqslant j \leqslant m$ to map $X \times Y$ into Z is that $S(f^*, g^*)$ belong to Z whenever f belongs to X and g belongs to Y. As we shall see, however, the most important applications start with the "extreme" estimates.

It would be interesting to find the proper definition of S so that necessary and sufficient conditions can be easily formulated in these cases as is the case for linear operators [13]. This would simplify considerably the extension process and allow further applications for non-separable spaces such as the $M(X)$ spaces. (1)

Remark 2.5. We should mention that the spaces $\Lambda(X)$ and $M(X)$ appear in [11] as K_A and M_A, where X is the Orlicz space determined by the Young's function A.

§ 3. Multilinear Stein–Weiss theory. In this section we derive some elementary intermediate results and use these to obtain the multilinear Stein–Weiss theorem with m initial conditions for the spaces $\Lambda_a(X)$. The function

$$F(r, s; t) = \frac{\varphi_Z(t)}{\varphi_X(r)\varphi_Y(s)}\Psi(r, s; t) \tag{3.1}$$

will be important in our considerations as we will be making L^p estimates on $T(f, g)^{**}(t)\varphi_Z(t)$ and

$$T(f, g)^{**}(t)\varphi_Z(t) \leqslant C\int_0^\infty\int_0^\infty [f^*(r)\varphi_X(r)][g^*(s)\varphi_Y(s)]F(r, s; t)\frac{dr}{r}\frac{ds}{s} \tag{3.2}$$

holds by inequality (2.9). We shall call the initial m triples of spaces $([X(j), Y(j)], Z(j))$, $1 \leqslant j \leqslant m$, an *interpolation segment* σ, and say that a *triple of spaces* $([X, Y], Z)$ *belong to* $\mathscr{W}(\sigma)$ if the operator S for σ maps $X \times Y$ into Z. Having established this notation, we can now catalogue several intermediate results of an elementary nature:

THEOREM 3.1. *If X, Y, and Z belong to $\mathscr{U} \cap \mathscr{L}$, then*

(i) $([M(X), \Lambda(Y)], M(Z)) \in \mathscr{W}(\sigma)$ *iff*

$$M_1 = \sup_{s,t}\left(\int_0^\infty F(r, s; t)\frac{dr}{r}\right) < \infty; \tag{3.3}$$

(ii) $([\Lambda(X), M(Y)], M(Z)) \in \mathscr{W}(\sigma)$ *iff*

$$M_2 = \sup_{r,t}\left(\int_0^\infty F(r, s; t)\frac{ds}{s}\right) < \infty; \tag{3.4}$$

(1) The author has since obtained a proof of Theorem 2.2. which does not require the separability of X or Y. This allows to drop the condition $\max(\alpha, \beta) > 0$ in Corollaries 4.7 and 4.8 below.

(iii) $([\Lambda(X), \Lambda(Y)], \Lambda(Z)) \in \mathscr{W}(\sigma)$ *iff*

$$M = \sup_{r,s}\left(\int_0^\infty F(r, s; t)\frac{dt}{t}\right) < \infty; \tag{3.5}$$

(iv) $([\Lambda(X), M(Y)], \Lambda(Z)) \in \mathscr{W}(\sigma)$ *iff*

$$N_1 = \sup_{r}\left(\int_0^\infty\int_0^\infty F(r, s; t)\frac{ds}{s}\frac{dt}{t}\right) < \infty; \tag{3.3'}$$

(v) $([M(X), \Lambda(Y)], \Lambda(Z)) \in \mathscr{W}(\sigma)$ *iff*

$$N_2 = \sup_{s}\left(\int_0^\infty\int_0^\infty F(r, s; t)\frac{dr}{r}\frac{dt}{t}\right) < \infty; \tag{3.4'}$$

(vi) $([M(X), M(Y)], M(Z)) \in \mathscr{W}(\sigma)$ *iff*

$$N = \sup_{t}\left(\int_0^\infty\int_0^\infty F(r, s; t)\frac{dr}{r}\frac{ds}{s}\right) < \infty; \tag{3.5'}$$

(vii) $([M(X), M(Y)], \Lambda(Z)) \in \mathscr{W}(\sigma)$ *iff* $\int\int\int F(r, s; t)\frac{dr}{r}\frac{ds}{s}\frac{dt}{t} < \infty$;

(viii) $([\Lambda(X), \Lambda(Y)], M(Z)) \in \mathscr{W}(\sigma)$ *iff* $\sup_{r,s,t} F(r, s, t) < \infty$.

Proof. It is clear that at the expense of brevity of notation, we could shorten the statement of this theorem using mixed norms. We only prove (i), since it is typical of the estimates used. Suppose f belongs to $M(X)$ and g belongs to $\Lambda(Y)$; then

$$\|S(f, g)\|_{M(Z)} \leqslant \|S(f^*, g^*)\|_{M(Z)}$$
$$\leqslant \sup_r \big(f^*(r)\varphi_X(r)\big)\sup_t\int\int g^*(s)\varphi_Y(s)F(r, s; t)\frac{dr}{r}\frac{ds}{s}$$

where the first inequality follows from the fact that $\left|\int h(s)k(s)ds\right| \leqslant \int_0^\infty h^*(s)k^*(s)ds$ applied to $\Psi(r, s; t)/(rs)$ which is decreasing in both r and s, and the second inequality comes from (3.1) and the definition of S. Since g belongs to $\Lambda(Y)$, we have

$$\|S(f, g)\|_{M(Z)} \leqslant \|f\|_{M(X)}\int g^*(s)\varphi_Y(s)\frac{ds}{s}\sup_{s,t}\int F(r, s; t)\frac{dr}{r}$$
$$\leqslant c\|f\|_{M(X)}\|g\|_{\Lambda(Y)}M_1.$$

Using these estimates, we can now formulate

THEOREM 3.2. *Suppose T is a bilinear operator of weak types $([X(j), Y(j)], Z(j))$, $1 \leqslant j \leqslant m$, and the rearrangement invariant spaces X, Y, and*

Z have fundamental functions so that the corresponding function F satisfies (3.3), (3.4), *and* (3.5), *then T has a unique extension so that*

$$\|T(f,g)\|_{\Lambda_\gamma(Z)} \leqslant CM_1^{\alpha'}M_2^{\beta'}M^\gamma \|f\|_{\Lambda_\alpha(X)}\|g\|_{\Lambda_\beta(Y)} \tag{3.6}$$

where $C=\max_{1\leqslant j\leqslant m} C_j$, C_j *the weak type* $([X(j), Y(j)], Z(j))$ *norm of* T, $\alpha+\beta \geqslant \gamma+1$, $\alpha'=1-\alpha$, $\beta'=1-\beta$, *and* $0\leqslant \alpha,\ \beta\leqslant 1$.

Proof. By the embedding $\Lambda_a(X)\subseteq \Lambda_b(X)$ when $a\geqslant b$, we may assume $\alpha+\beta=\gamma+1$. We may also assume $\gamma>0$ even though the following proof is still valid in this case with simplified estimates at each stage. From $\alpha+\beta=\gamma+1$ and $\gamma>0$, then α and β will both be positive. Now we let $a=\gamma/\alpha$ and $b=\gamma/\beta$, so $0<a,\ b\leqslant 1$. Letting $a'=1-a=\beta'/\alpha$ and $b'=1-b=\alpha'/\beta$ (with the standard modifications if α' or β' is zero), we have

$$\begin{aligned}
&S(f^*,g^*)(t)\varphi_Z(t)\\
&= \int_0^\infty\int_0^\infty (f^*\varphi_X)^{a'}(r)\,[(f^*\varphi_X)^{a}(r)(g^*\varphi_Y)^{b}(s)]\,(g^*\varphi_Y)^{b'}(s)F(r,s;t)\frac{dr}{r}\frac{ds}{s}\\
&\leqslant \Big(\int_0^\infty\int_0^\infty (f^*\varphi_X)^{1/\alpha}(r)F(r,s;t)\frac{dr}{r}\frac{ds}{s}\Big)^{\beta'}\times\\
&\qquad \times\Big(\int_0^\infty\int_0^\infty (f^*\varphi_X)^{a/\gamma}(r)(g^*\varphi_Y)^{b/\gamma}(s)F(r,s;t)\frac{dr}{r}\frac{ds}{s}\Big)^{\gamma}\times\\
&\qquad \times\Big(\int_0^\infty\int_0^\infty (g^*\varphi_Y)^{1/\beta}(s)F(r,s;t)\frac{dr}{r}\frac{ds}{s}\Big)^{\alpha'}
\end{aligned}$$

by applying Hölder's inequality with $\beta'+\gamma+\alpha'=1$ and weight $F(r,s;t)/(rs)$, so

$$S(f^*,g^*)(t)\varphi_Z(t)\leqslant$$
$$M_2^{\beta'}\|f\|_{\Lambda_\alpha(X)}^{\beta'/\alpha}\Big[\int_0^\infty\int_0^\infty (f^*\varphi_X)^{1/\alpha}(g^*\varphi_Y)^{1/\beta}F\frac{dr}{r}\frac{ds}{s}\Big]^{\gamma}M_1^{\alpha'}\|g\|_{\Lambda_\beta(Y)}^{\alpha'/\beta}.$$

Taking the $L^{1/\gamma}$ norm, we have by (2.9)

$$\begin{aligned}
\|T(f,g)\|_{\Lambda_\gamma(Z)} &\leqslant CM_1^{\alpha'}M_2^{\beta'}\|f\|_{\Lambda_\alpha(X)}^{a'}\|g\|_{\Lambda_\beta(Y)}^{b'}\times\\
&\times\Big\{\int_0^\infty\int_0^\infty\int_0^\infty (f^*\varphi_X)^{1/\alpha}(r)(g^*\varphi_Y)^{1/\beta}(s)F(r,s;t)\frac{dr}{r}\frac{ds}{s}\frac{dt}{t}\Big\}^{\gamma}.
\end{aligned}\tag{3.7}$$

But by Fubini's theorem,

$$\Big\{\int_0^\infty\int_0^\infty\int_0^\infty (f^*\varphi_X)^{1/\alpha}(r)(g^*\varphi_Y)^{1/\beta}(s)F(r,s;t)\frac{dr}{r}\frac{ds}{s}\frac{dt}{t}\Big\}^{\gamma}\leqslant \|f\|_{\Lambda_\alpha(X)}^{a}\|g\|_{\Lambda_\beta(Y)}^{b}M^\gamma;$$

but this together with (3.7) implies

$$\|T(f,g)\|_{\Lambda_\gamma(Z)}\leqslant CM_1^{\alpha'}M_2^{\beta'}M^\gamma\|f\|_{\Lambda_\alpha(X)}\|g\|_{\Lambda_\beta(Y)}.$$

It should be noted here that this proof is just a modified version of a proof given by O'Neil (Lemma (10.1) of [11]) for integral operators on $L^{p,q}$ spaces.

Remark 3.3. If for the classicial interpolation scheme of L^p spaces we consider

$$\Phi(u,v;t)=\min_{j=1,2}\Big\{C_j\frac{u^{1/p_j}v^{1/q_j}}{t^{1/r_j}}\Big\}$$

instead of the function $\Psi(u,v;t)$, we can obtain the following refinement of the intermediate operator bound

$$\begin{aligned}
&\|T(f,g)\|_{L^{r,1/\gamma}}\\
&\leqslant \frac{C_2^\theta C_1^{1-\theta}}{\theta(1-\theta)}\Big(\frac{p_1p_2}{|p_1-p_2|}\Big)^{\alpha'}\Big(\frac{q_1q_2}{|q_1-q_2|}\Big)^{\beta'}\Big(\frac{r_1r_2}{|r_1-r_2|}\Big)^{\gamma}\|f\|_{L^{p,1/\alpha}}\|g\|_{L^{q,1/\beta}},
\end{aligned}\tag{3.8}$$

where $\Big(\frac{1}{p},\frac{1}{q},\frac{1}{r}\Big)=(1-\theta)\Big(\frac{1}{p_1},\frac{1}{q_1},\frac{1}{r_1}\Big)+\theta\Big(\frac{1}{p_2},\frac{1}{q_2},\frac{1}{r_2}\Big)$ and C_j is the weak type operator bound. Therefore, by evaluating the modified M_1, M_2, and M of Theorem 3.1, we are able to obtain the full power of the classical theorem.

In order to state the general multilinear theorem we need to develop some notation. A multilinear operator is called *weak type* $([X(1,j),\ldots$ $\ldots, X(n,j)], Z(j))$ *with norm* C_j if

$$T(\chi_{E_1},\ldots,\chi_{E_n})^{**}(t)\varphi_{Z(j)}(t)\leqslant C_j\prod_{i=1}^n\varphi_{X(i,j)}(mE_i) \tag{3.9}$$

and C_j is the smallest constant for which (3.9) holds independent of the sets E_i and t. We define

$$\Psi(\mathbf{s};t)=\min_{1\leqslant j\leqslant m}\left(\frac{\prod_{i=1}^n\varphi_{X(i,j)}(s_i)}{\varphi_{Z(j)}(t)}\right),$$

where $\mathbf{s}=(s_1,\ldots,s_n)$. If we let

$$F(\mathbf{s};t)=\frac{\varphi_Z(t)}{\prod_{i=1}^n\varphi_{X(i)}(s_i)}\Psi(\mathbf{s};t)$$

and then set

$$M_i = \sup_{s_k(k\neq i),t} \int_0^\infty F(\mathbf{s};t)\frac{ds_i}{s_i} \quad \text{and} \quad M = \sup_{s_k} \int_0^\infty F(\mathbf{s};t)\frac{dt}{t},$$

we can prove the following theorem using the same techniques as above.

THEOREM 3.4. *If T is a multilinear operator of weak types $\big([X(1,j),\ldots,X(n,j)], Z(j)\big)$, $1\leqslant j\leqslant m$, then T has a unique extension so that*

$$\|T(f_1,\ldots,f_n)\|_{\Lambda_\gamma(Z)} \leqslant M^\gamma \prod_{i=1}^n (M_i^{a_i'}\|f\|_{\Lambda_{a_i}(X(i))})$$

whenever $\sum_{i=1}^n a_i \geqslant n-1+\gamma$.

Remark 3.5. M. Zafran [19] has generalized the multilinear Stein–Weiss theorem in another direction using the Peetre K-theory and has obtained very nice applications of a different nature. Weak type interpolation with m initial estimates was established in [14] for linear operators. Strong type interpolation of m pairs was carried out for Peetre's theory in [16] and [18]. It appears that this paper is the first attempt at mixing these ideas.

Remark 3.6. It is not hard to see that a necessary condition for $\big([X(i),\ldots,X(n)], Z\big)$ to be weak intermediate for the interpolation segment of L^p spaces with indices $\big([p(1,j),\ldots,p(n,j)], q(j)\big)$, $j=1,2$, is that the X's and Z must satisfy

$$\Big(\prod_{i=1}^n s_i^{b_i}\Big)\varphi_Z\Big(\prod_{i=1}^n s_i^{m_i}\Big) \leqslant c\prod_{i=1}^n \varphi_{X(i)}(s_i) \qquad \text{all } s_i,$$

where $m_i = \dfrac{a(i,1)-a(i,2)}{\beta_1-\beta_2}$, $b_i = a(i,1)-m_i\beta_1$, $a(i,j) = \dfrac{1}{p(i,j)}$, and $\beta_j = \dfrac{1}{q_j}$, $i=1,\ldots,n$, $j=1,2$. For the case of linear operators this reduces to

$$s^b\varphi_Z(s^m) \leqslant c\varphi_X(s) \qquad \text{all } s,$$

where $m = \dfrac{a_1-a_2}{\beta_1-\beta_2}$, $b = a_1-m\beta_1$. Using these conditions one can easily give sufficient conditions for weak interpolation ([4], [17]).

One can obtain slightly more general theorems if the endpoint spaces have fundamental functions which are compatible in the sense that they behave like powers [17]. Although the weak type results proved in [17] follow as special cases of the results presented here (actually of the results in [13]), Torchinsky considers mixed weak type and strong type theorems, as well as strong type theorems, which generalize the classical results in a direction that allows recovery of endpoint estimates.

Next we state a multilinear interpolation theorem of a different character than Theorem 3.2 which will prove useful in the next section. The proof of the following theorem and its generalization for multilinear operators (which we leave to the interested reader to formulate) is a simple exercise in multiple applications of Hölder's inequality.

THEOREM 3.7. *Suppose T is a bilinear operator of weak types $\big([X(j), Y(j)], Z(j)\big)$, $1\leqslant j\leqslant m$, and the spaces X, Y, and Z have fundamental functions so that the corresponding function F satisfies conditions* (3.3′), (3.4′), *and* (3.5′), *then T has a unique extension so that*

$$\|T(f,g)\|_{\Lambda_\gamma(Z)} \leqslant CN_1^\alpha N_2^\beta N^{\gamma'}\|f\|_{\Lambda_\alpha(X)}\|g\|_{\Lambda_\beta(Y)}$$

if $\alpha+\beta\geqslant\gamma$ and $\max(\alpha,\beta)>0$, where $C = \max_{1\leqslant j\leqslant m} C_j^3$ and C_j is the weak type $\big([X(j), Y(j)], Z(j)\big)$ operator norm.

§ 4. Applications. In this section we apply the results of Section 3 to specific bilinear operators: integral operators, tensor product operators, convolution, and product operators. Most of the results of this section have been obtained for $L^{p,q}$ spaces by O'Neil ([9], [11]) and for $\Lambda_a(X)$ spaces in [7] and [8]. The general approach of interpolation taken here for these operators seems new.

A bilinear operator I is called an *integral operator* if it satisfies the estimates: $\|I(f,g)\|_1\leqslant\|f\|_1\|g\|_\infty$ and $\|I(f,g)\|_\infty\leqslant\|f\|_\infty\|g\|_1$. The primary example of an integral operator is given by $I(f,g)(t) = \int f(s,t)g(s)\,ds$. The interpolation scheme for these operators is $([L^1, L^\infty], L^1)$ and $([L^\infty, L^1], L^\infty)$ and so in this case $\Psi_I(r,s;t) = \min(r/t, s)$. By Theorem (2.1), we have the basic inequality for integral operators:

LEMMA 4.1. *If I is an integral operator, then*

$$I(f,g)^{**}(t) \leqslant S_I(f^*,g^*)(t) = \int_0^\infty [(P+Q)f^*](st)g^*(s)\,ds. \tag{4.1}$$

This should be compared with the basic inequality derived in [11]

$$I(f,g)^{**}(t) \leqslant \int_0^\infty f^*(st)g^*(s)\,ds.$$

LEMMA 4.2. *A necessary condition that integral operators map $X\times Y$ into Z is that the "fundamental condition for integral operators"*

$$a\varphi_Z(b) \leqslant \text{const}\,\varphi_X(ab)\varphi_Y(a) \qquad \textit{all } a, b \tag{4.2}$$

hold.

Proof. If we consider the integral operator $I(f,g)(t) = \int f(s,t)g(s)\,ds$ with $f_{..}(s,t) = \chi_{(0,a)}(s)\chi_{(0,b)}(t)$ and $g_a(s) = \chi_{(0,a)}(s)$, then if I maps $X\times Y$

into Z, we have

$$\sup_{f\in X, g\in Y}\left(\sup_t \frac{I(f,g)^*(t)\varphi_Z(t)}{\|f\|_X\|g\|_Y}\right) \leqslant \sup_{f\in X, g\in Y}\frac{\|I(f,g)\|_Z}{\|f\|_X\|g\|_Y} \leqslant \text{const}.$$

But $I(f_{a,b}, g_a)^* = a\chi_{(0,b)}, f^*_{a,b} = \chi_{(0,ab)}$, and $g^*_a = g_a$, so

$$\sup_{a,b}\left(\frac{a\varphi_Z(b)}{\varphi_X(ab)\varphi_Y(a)}\right) \leqslant \text{const}.$$

It is not hard to show that condition (4.2) is necessary and sufficient in order for relations (3.3), (3.4), and (3.5) to hold. Typical is relation (3.3),

$$\sup_{s,t}\left(\int_0^\infty F_I(r,s;t)\frac{dr}{r}\right) = \sup_{s,t}\left(\int_0^{st}\frac{\varphi_Z(t)}{\varphi_X(r)\varphi_Y(s)}\frac{r}{t}\frac{dr}{r} + \int_{st}^\infty \frac{\varphi_Z(t)s}{\varphi_X(r)\varphi_Y(s)}\frac{dr}{r}\right)$$

$$\leqslant \sup_{s,t}\left\{\frac{s\varphi_Z(t)}{\varphi_Y(s)}\left(\frac{1}{st}\int_0^{st}\varphi_{X'}(r)\frac{dr}{r} + \int_{st}^\infty \frac{1}{\varphi_X(r)}\frac{dr}{r}\right)\right\}.$$

But by relations (2.4) and (2.5), we have

$$\sup_{s,t}\int^\infty F_I(r,s;t)\frac{dr}{r} \leqslant \text{const}\sup_{s,t}\frac{s\varphi_Z(t)}{\varphi_X(st)\varphi_Y(s)} \leqslant \text{const}.$$

Relations (3.4) and (3.5) follow similarly, so by Theorem 3.2 we have

Corollary 4.3. *If X, Y, and Z belong to $\mathcal{U}\cap\mathcal{L}$ and I is an integral operator, then I has a unique extension so that*

$$\|I(f,g)\|_{\Lambda_\gamma(Z)} \leqslant \text{const}\,\|f\|_{\Lambda_\alpha(X)}\|g\|_{\Lambda_\beta(Y)}$$

if $\alpha+\beta\geqslant\gamma+1$ and condition (4.2) holds.

Closely connected with integral operators are *tensor product operators*, i.e., bilinear operators which satisfy

$$\|T(f,g)\|_\infty \leqslant \|f\|_\infty\|g\|_\infty \quad\text{and}\quad \|T(f,g)\|_1 \leqslant \|f\|_1\|g\|_1.$$

An example of such an operator is, of course, $T(f,g)(r,s) = (f\otimes g)(r,s) = f(r)g(s)$. In this scheme we have $\Psi_T(r,s;t) = \min(rs/t, 1)$ and hence the "basic inequality for tensor products" by Theorem 2.1 is computed to be

Lemma (4.4). *If T is a tensor product operator, then*

$$T(f,g)^{**}(t) \leqslant \int_0^\infty [(P+Q)f^*](t/s)g^*(s)\frac{ds}{s}. \tag{4.3}$$

Lemma (4.5). *The "fundamental condition for tensor products"*

$$\varphi_Z(ab) \leqslant \text{const}\,\varphi_X(a)\varphi_Y(b) \quad \text{all } a, b \tag{4.4}$$

is necessary in order that T map $X\times Y$ into Z.

Proof. If $T(f,g) = f\otimes g$ and we let $f_a = \chi_{(0,a)}, g_b = \chi_{(0,b)}$, then $T(f_a, g_b)^* = \chi_{(0,ab)}$ and we obtain

$$\sup_{a,b}\frac{\varphi_Z(ab)}{\varphi_X(a)\varphi_Y(b)} = \sup_{a,b}\sup_t\left(\frac{T(f_a,g_b)^*(t)\varphi_Z(t)}{\varphi_X(a)\varphi_Y(b)}\right)$$

$$\leqslant \sup_{f\in X, g\in Y}\frac{\|T(f,g)\|_Z}{\|f\|_X\|g\|_Y} \leqslant \text{const}.$$

As before we can show that condition (4.4) is necessary and sufficient for $F_T(r,s;t) = \frac{\varphi_Z(t)}{\varphi_X(r)\varphi_Y(s)}\min(rs/t, 1)$ to satisfy (3.3) through (3.5), and so by Theorem 3.2 we have

Corollary 4.6. *If X, Y, and Z belong to $\mathcal{U}\cap\mathcal{L}$ and satisfy the "fundamental condition for tensor products" (4.4), and T is a tensor product operator, then T has a unique extension so that*

$$\|T(f,g)\|_{\Lambda_\gamma(Z)} \leqslant \text{const}\,\|f\|_{\Lambda_\alpha(X)}\|g\|_{\Lambda_\beta(Y)}$$

if $\alpha+\beta\geqslant\gamma+1$.

A bilinear operator C which satisfies the three estimates $\|C(f,g)\|_1 \leqslant \|f\|_1\|g\|_1$, $\|C(f,g)\|_\infty \leqslant \|f\|_\infty\|g\|_1$, and $\|C(f,g)\|_\infty \leqslant \|f\|_1\|g\|_\infty$ is called a *convolution operator* ([9], [11], [7]). $C(f,g)(t) = (f*g)(t) = \int f(t-s)g(s)\,ds$ is the common example of a convolution operator. Here the appropriate Ψ is $\Psi_C(r,s;t) = \min(r,s,rs/t)$. As before we can compute the "basic inequality for convolution"

$$C(f,g)^{**}(t) \leqslant S_c(f^*,g^*)(t) = tg^{**}(t)[(P+Q)(f^*)](t) + \int_t^\infty [(P+Q)f^*](s)g^*(s)\,ds. \tag{4.5}$$

Compare this with the inequality in [9]

$$C(f,g)^{**}(t) \leqslant tg^{**}(t)f^{**}(t) + \int_t^\infty f^*(s)g^*(s)\,ds. \tag{4.6}$$

As before, the inequality (4.5) suffices to estimate convolution operators on the interior of an interpolation segment.

Since $(\chi_{(0,a)}*\chi_{(0,a)})^*(t) = (a-t/2)\chi_{(0,2a)}(t)$, we see by letting $f = g = \chi_{(0,a)}$ that in order for

$$\sup_{f\in X, g\in Y}\left(\sup_t\frac{(f*g)^*(t)\varphi_Z(t)}{\|f\|_X\|g\|_Y}\right) \leqslant \text{const}$$

it is necessary that $\sup_a \frac{a\varphi_Z(a)}{\varphi_X(a)\varphi_Y(a)} \leqslant \text{const}$, i.e.

$$a\varphi_Z(a) \leqslant \text{const}\,\varphi_X(a)\varphi_Y(a) \quad \text{all } a. \tag{4.7}$$

The "fundamental condition for convolution" (4.7) is necessary and sufficient in order for conditions (3.3′) through (3.5′) to hold for $F = F_c$. In fact, we now have, by Theorem 3.7,

COROLLARY 4.7. *Suppose X, Y, and Z belong to $\mathscr{U} \cap \mathscr{L}$. If C is a convolution operator and condition* (4.7) *holds, then C has a unique extension so that*

$$\|C(f, g)\|_{\Lambda_\gamma(Z)} \leqslant \text{const} \, \|f\|_{\Lambda_\alpha(X)} \|g\|_{\Lambda_\beta(Y)}$$

if $\alpha+\beta \geqslant \gamma$ *and* $\max(\alpha, \beta) > 0$.

Intimately related to convolution operators are the *product operators*. These are bilinear operators which satisfy the conditions $\|P(f, g)\|_\infty \leqslant \|f\|_\infty \|g\|_\infty$, $\|P(f, g)\|_1 \leqslant \|f\|_1 \|g\|_\infty$, and $\|P(f, g)\|_1 \leqslant \|f\|_\infty \|g\|_1$ (see [9]). The common example of a product operator is $P(f, g) = fg$. The Ψ in this case is $\Psi_P(r, s; t) = \min\left(\frac{r}{t}, \frac{s}{t}, 1\right)$.

By letting $f_a = \chi_{(0,a)} = g_a$ and $P(f, g) = fg$, we may easily show that a necessary condition for P to map $X \times Y$ into Z is the "fundamental condition for product operators"

$$\varphi_Z(a) \leqslant \text{const} \, \varphi_X(a) \varphi_Y(a) \quad \text{all } a. \tag{4.8}$$

By computing $S_P(f^*, g^*)(t)$, we can derive the "basic inequality for product operators"

$$P(f, g)^{**}(t) \leqslant \frac{1}{t} \int_0^t [g^*(Qf^*) + f^*(Qg^*)] \, ds + (Qf^*)(t)(Qg^*)(t).$$

It is not difficult to show that (4.8) is necessary and sufficient for conditions (3.3′) through (3.5′) to hold for F_P. Proceeding exactly as in the case of convolution operators, we apply Theorem 3.7 to get

COROLLARY 4.8. *Suppose X, Y, and Z belong to $\mathscr{U} \cap \mathscr{L}$ and P is a product operator, then P has a unique extension so that*

$$\|P(f, g)\|_{\Lambda_\gamma(Z)} \leqslant \text{const} \, \|f\|_{\Lambda_\alpha(X)} \|g\|_{\Lambda_\beta(Y)}$$

whenever $\alpha+\beta \geqslant \gamma$, $\max(\alpha, \beta) > 0$, *and X, Y, and Z satisfy condition* (4.8).

§ 5. Final remarks. In this section we briefly remark how the techniques used in this paper and [13] may be used in the case of linear operators. It is well-known that the Calderón theory of weak interpolation may be applied to many of the standard linear and sublinear operators of Fourier analysis such as the Hilbert transform [2], Hardy–Littlewood maximal function [2], Fourier transform [1], Laplace transform [6], and fractional integrals [15]. We illustrate the power of the these techniques

by concentrating on the real Laplace transform given by

$$\mathscr{L}f(t) = \int_0^\infty f(s) e^{-st} ds, \quad t > 0.$$

It is easy to see that $\mathscr{L}$ must be a bounded operator from L^1 to L^∞ and L^∞ to L^1. In this case $\Psi(s, t) = \min(s, 1/t)$, and therefore the basic inequality (from [13]) is

$$\mathscr{L}(f)^*(t) \leqslant 2 \frac{1}{t} f^{**}\left(\frac{1}{t}\right). \tag{5.1}$$

Since $\mathscr{L}(\chi_{(0,a)})^*(t) = (1-e^{-at})/t$, then a necessary condition for $\mathscr{L}$ to map X to Y is that the "fundamental condition"

$$a\varphi_Y(1/a) \leqslant \text{const} \, \varphi_X(a), \quad \text{all } a \tag{5.2}$$

holds. In fact,

$$\begin{aligned}(1-e^{-1}) \sup_a \frac{\varphi_Y(1/a)\, a}{\varphi_X(a)} &\leqslant \sup_a \left(\sup_t \frac{1-e^{-at}}{t} \frac{\varphi_Y(t)}{\varphi_X(a)}\right) \\ &\leqslant \sup_a \frac{\|\mathscr{L}\chi_{(0,a)}\|_{M(Y)}}{\|\chi_{(0,a)}\|_X} \\ &\leqslant \sup_{f \in X} \frac{\|\mathscr{L}f\|_Y}{\|f\|_X} \leqslant \text{const}.\end{aligned}$$

Condition (5.2) implies that both $\sup_t \int_0^\infty F(s, t) \frac{ds}{s}$ and $\sup_t \int_0^\infty F(s, t) \frac{dt}{t}$ are finite, so $\mathscr{L}$ has a unique extension so that

$$\|\mathscr{L}(f)\|_{\Lambda_\alpha(Y)} \leqslant \text{const} \, \|f\|_{\Lambda_\alpha(X)}, \quad 0 \leqslant \alpha \leqslant 1.$$

This theorem generalizes Corollary (10.12) of [11] and was first obtained in [6].

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Equivalence of Haar and Franklin bases in L_p spaces*

by

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Abstract. The main result of this paper states that the Haar and Franklin orthonormal sets do form equivalent bases in $L_p\langle 0,1\rangle$ for each p, $1<p<\infty$, i.e. the spaces of coefficients for the two bases are identical. The proof depends on the unconditionality of Haar and Franklin bases. The original proof of S. V. Bockariev of the unconditionality of the Franklin basis is rather complicated and a simplified version is presented in this paper. As a consequence of our main result we obtain the L_p version of the maximal inequality for the Fourier partial sums of the uniformly bounded orthonormal system of polygonals introduced earlier by one of the authors.

1. Introduction. In his recent paper S. V. Bockariev [1] (see also [2]) proved that the Franklin system is an unconditional basis in $L_p(I)$, $I=\langle 0,1\rangle$, $1<p<\infty$. His ingenious proof reguiies only, appart from the properties of the Franklin functions established by Z. Ciesielski in [4], a modification of the A. Zygmund lemma on decomposition of functions and the weak type L_1 estimate for the Hardy–Littlewood maximal function. It appears that with the help of some technics known in the theory of singular integrals the original proof of Bockariev can be modified considerably. Such a simplified version of the proof of unconditionality of the Franklin basis is presented below, and the additional tools used in it are the Whitney's decomposition lemma of open sets into dyadic cubes, the Marcinkiewicz integral and distance function.

The unconditionality of the Franklin basis (S. V. Bockariev) and of the Haar basis (J. Marcinkiewicz) in L_p, $1<p<\infty$, is the starting point in the proof of our main result, i.e. the equivalence of the Haar and Franklin bases in L_p, $1<p<\infty$. The essential step leading to the desired result is an application of the Fefferman–Stein inequality (Theorem E) to both Haar and Franklin systems. To do this, it is necessary to compare maximal functions of the function belonging to one system with the corresponding functions from the other system. However, this can be done on the basis of the estimate for Franklin functions obtained by Z. Ciesielski in [4].

* It has been proved recently by P. Sjölin that the Haar and Franklin bases are not equivalent in L_1.