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Hardy spaces associated with non-negative self-adjoint operators

by

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Abstract. The maximal and atomic Hardy spaces H^p and H^p_A, $0 < p \le 1$, are considered in the setting of a doubling metric measure space in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization. It is shown that $H^p = H^p_A$ with equivalent norms.

1. Introduction. The purpose of this article is to establish the equivalence of the maximal and atomic Hardy spaces H^p and H^p_A, $0 < p \le 1$, in the general setting of a metric measure space with the doubling property and in the presence of a non-negative self-adjoint operator L whose heat kernel has Gaussian localization. We next describe our setting in detail:

I. We assume that (X, ρ, μ) is a metric measure space such that (X, ρ) is a locally compact metric space with distance $\rho(\cdot,\cdot)$ and μ is a positive Radon measure. We also stipulate the *volume doubling condition*:

$$0 < \mu(B(x, 2r)) \le c_0 \mu(B(x, r)) < \infty \quad \text{for all } x \in X \text{ and } r > 0, \tag{1.1}$$

where $B(x, r)$ is the open ball centered at x of radius r and $c_0 > 1$ is a constant. It follows that

$$\mu(B(x, \lambda r)) \le c_0 \lambda^d \mu(B(x, r)) \quad \text{for } x \in X,\ r > 0, \text{ and } \lambda > 1, \tag{1.2}$$

where $d = \log_2 c_0 > 0$ is a constant playing the role of dimension.

II. The main assumptions are:

(H1) L is a non-negative self-adjoint operator on $L^2(X, d\mu)$, mapping real-valued to real-valued functions.

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(H2) The semigroup $P_t = e^{-tL}$, $t > 0$, associated with L consists of integral operators with (heat) kernel $p_t(x, y)$ having a Gaussian upper bound, that is, $p_t(x, y)$ is a measurable function on $X \times X$ and there exists a set $\widetilde{X} \subset X$, independent of t, with $\mu(X \setminus \widetilde{X}) = 0$ such that

$$|p_t(x,y)| \le \frac{C^\star \exp\{-c^\star \rho^2(x,y)/t\}}{[\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))]^{1/2}}, \quad \forall x,y \in \widetilde{X},\ \forall t > 0. \tag{1.3}$$

Above, $C^\star, c^\star > 0$ are structural constants.

The definitions of the maximal and atomic Hardy spaces in the setting described above will be based on $L^2(X)$ and will follow in the footsteps of [5, 6, 4].

Definition 1.1. The *maximal Hardy space* H^p, $0 < p \le 1$, in the general setting described above is defined as the completion of the set of all functions $f \in L^2(X)$ such that

$$\|f\|_{H^p} := \Big\| \sup_{t>0} |e^{-t^2 L} f(\cdot)| \Big\|_{L^p} < \infty.$$

Just as in the classical case on $\mathbb{R}^n$, non-tangential, tangential, and grand maximal operators will be introduced and the equivalence of $\|f\|_{H^p}$ with quasi-norms defined by the respective maximal operators will be established.

We consider two versions of atomic Hardy spaces in the current setting, depending on whether $\mu(X) = \infty$ or $\mu(X) < \infty$.

Atomic Hardy spaces in the case $\mu(X) = \infty$

Definition 1.2. Let $0 < p \le 1$ and $n := \lfloor d/2p \rfloor + 1$, where d is from the doubling property (1.2). A function a is called an *atom* associated with the operator L if there exists a function $b \in D(L^n)$ and a ball B of radius $r = r_B > 0$ such that

(i) $a = L^n b$,
(ii) $\operatorname{supp} L^k b \subset B$, $k = 0, 1, \dots, n$, and
(iii) $\|L^k b\|_\infty \le r^{2(n-k)} |B|^{-1/p}$, $k = 0, 1, \dots, n$.

Definition 1.3. The *atomic Hardy space* H^p_A, $0 < p \le 1$, is defined as follows. We say that $f = \sum_{j \ge 1} \lambda_j a_j$ is an *atomic representation* of f if $\{\lambda_j\}_{j \ge 1} \in \ell^p$, all a_j, $j = 1, 2, \dots$, are atoms, and the series converges in L^2. We denote by $\mathbb{H}^p_A$ the space of all functions $f \in L^2(X)$ that have atomic representations with norm defined by

$$\|f\|_{\mathbb{H}^p_A} := \inf_{f = \sum_{j \ge 1} \lambda_j a_j} \Big(\sum_{j \ge 1} |\lambda_j|^p \Big)^{1/p}, \qquad f \in \mathbb{H}^p_A.$$

Now, H^p_A, $0 < p \le 1$, is defined as the completion of $\mathbb{H}^p_A$ with respect to the above norm.

Atomic Hardy spaces in the case $\mu(X) < \infty$. In this case we use the atoms from Definition 1.2 with the addition of one more kind of atoms, say $A \in L^\infty(X)$, with the property

$$\|A\|_\infty \le |X|^{-1/p}. \tag{1.4}$$

Then the atomic Hardy space H^p_A, $0 < p \le 1$, is defined just as in the case $\mu(X) = \infty$ above.

We now come to the main result in this article.

Theorem 1.4. *In the setting of this paper, we have $H^p = H^p_A$, $0 < p \le 1$, and*

$$\|f\|_{H^p_A} \sim \|f\|_{H^p} \quad \text{for } f \in H^p. \tag{1.5}$$

This result has been obtained in [11] in the setting of $\mathbb{R}^n$ in the presence of a non-negative self-adjoint operator L with heat kernel having Gaussian localization. The proof in [11] heavily relies on the geometry of $\mathbb{R}^n$ and is based on a technique due to A. Calderón [1].

To prove Theorem 1.4 we devise a new approach that is different from the one in [1, 11] as well as the classical proof that uses the Calderón–Zygmund decomposition.

Characterizations of atomic Hardy spaces via square functions and their molecular decompositions are obtained in [5] for H^1, in [4] for H^p, $0 < p \le 1$, and in [6] for Orlicz–Hardy spaces, in somewhat different settings. We shall not elaborate on these kind of results here.

This paper is organized as follows. In §2 we assemble the necessary background material from [2, 7]. In §3 we introduce the maximal Hardy spaces and establish their characterization via several maximal operators. In §4 we prove our main result: the equivalence of maximal and atomic Hardy spaces. Section 5 is an appendix where we place the proofs of some ancillary assertions from previous sections.

Notation. For an arbitrary set $E \subset X$ and $x \in X$ we shall denote $\operatorname{dist}(x, E) := \inf_{y\in E} \rho(x, y)$, $E^c := X \setminus E$, $|E| := \mu(E)$, and $\overline{E}$ is the closure of E. We shall abbreviate "almost all" by "a.a." and sup will stand for ess sup. The notation $cB(x, \delta) := B(x, c\delta)$ will be used. The class of Schwartz functions on $\mathbb{R}$ will be denoted by $\mathcal{S}(\mathbb{R})$. As usual, $C_0^\infty(\mathbb{R})$ will stand for the class of all compactly supported C^∞ functions on $\mathbb{R}$. Positive constants will be denoted by c, c_1, c', ... and they may vary at every occurrence. Most of them will depend on the basic structural constants $c_0, C^\star, c^\star$ from (1.1)–(1.3). Usually, this dependence will not be indicated explicitly. The notation $a \sim b$ will mean $c_1 \le a/b \le c_2$.

2. Background. Our development of Hardy spaces will rely on some basic facts and results from [2, 7], which we review next.

2.1. Inequalities related to the geometry of the underlying space. To compare the volumes of balls with different centers $x, y \in X$ and the same radius r we shall use the inequality

$$|B(x,r)| \le c_0(1+\rho(x,y)/r)^d |B(y,r)|, \quad x,y \in X,\ r>0. \tag{2.1}$$

As $B(x,r) \subset B(y,\rho(y,x)+r)$ the above inequality is immediate from (1.2).

The following simple inequalities will also be needed [7, Lemma 2.1]: for $\sigma > d$ and $t > 0$,

$$\int_X (1+t^{-1}\rho(x,y))^{-\sigma}\, d\mu(y) \le c|B(x,t)|, \quad x \in X, \tag{2.2}$$

2.2. Functional calculus. Observe that as L is a non-negative self-adjoint operator that maps real-valued to real-valued functions, for any real-valued, measurable and bounded function G on $\mathbb{R}_+$ the operator $G(L)$ defined by

$$G(L) := \int_0^\infty G(\lambda)\, dE_\lambda,$$

with E_λ, $\lambda \ge 0$, being the spectral resolution associated with L, is bounded on L^2, self-adjoint, and maps real-valued functions to real-valued functions.

The following *Davies–Gaffney estimate* follows from our basic assumptions I–II (see [3, 7]):

$$|\langle P_t f_1, f_2\rangle| \le \exp\{-c^\star r^2/t\}\|f_1\|_2\|f_2\|_2, \quad t>0, \tag{2.3}$$

for all open sets $U_j \subset X$ and $f_j \in L^2(X)$ with $\operatorname{supp} f_j \subset U_j$, $j=1,2$, where $r := \rho(U_1,U_2)$ and $c^\star > 0$ is the constant from (1.3).

In turn, the Davies–Gaffney estimate implies (see [3]) the *finite speed propagation property*, which will play a crucial role in our theory:

$$\langle \cos(t\sqrt{L})f_1, f_2\rangle = 0, \quad 0<\tilde{c}t<r,\ \tilde{c} := \frac{1}{2\sqrt{c^\star}}, \tag{2.4}$$

for all open sets $U_j \subset X$, $f_j \in L^2(X)$, $\operatorname{supp} f_j \subset U_j$, $j=1,2$, where $r := \rho(U_1,U_2)$.

The finite speed propagation property leads to the following localization result for the kernels of operators of the form $G(t\sqrt{L})$ whenever $\hat{G}$ is band limited. Here $\hat{G}(\xi) := \int_{\mathbb{R}} G(x)e^{-ix\xi}\,dx$.

Proposition 2.1. *Let G be even,* $\operatorname{supp}\hat{G} \subset [-A,A]$ *for some $A>0$, and $\hat{G} \in W_1^m$ for some $m>d$, i.e. $\|\hat{G}^{(m)}\|_{L^1} < \infty$. Then for any $t>0$ and $x,y \in X$,*

$$G(t\sqrt{L})(x,y) = 0 \quad \textit{if } \rho(x,y) > \tilde{c}tA. \tag{2.5}$$

This assertion follows from [5, proof of Lemma 3.5].

The next proposition is another important ingredient in establishing our basic localization result for the kernels of operators of the form $\varphi(\sqrt{L})$ for smooth functions φ.

PROPOSITION 2.2. *Let G be a bounded measurable function on $\mathbb{R}_+$ with $\operatorname{supp} G \subset [0,\tau]$ for some $\tau > 0$. Then $G(\sqrt{L})$ is an integral operator with kernel $G(\sqrt{L})(x,y)$ satisfying*

$$(2.6) \qquad |G(\sqrt{L})(x,y)| \le \frac{c_\flat \|G\|_\infty}{(|B(x,\tau^{-1})|\,|B(y,\tau^{-1})|)^{1/2}}, \qquad \forall x,y \in \widetilde{X},$$

where $\widetilde{X} \subset X$ with $\mu(X \setminus \widetilde{X}) = 0$ is from (1.3) *and $c_\flat > 0$ depends only on the constants $c_0, C^\star, c^\star$ from* (1.1), (1.3).

Proof. This result is essentially contained in [2, Theorem 3.7]. We next present the argument in order to show that it is independent of the additional assumptions in [2].

Under the above hypothesis, clearly $G(\sqrt{L}) = e^{-t^2L}[e^{2t^2L}G(\sqrt{L})]e^{-t^2L}$ for all $t > 0$, and

$$\|e^{2t^2L}G(\sqrt{L})\|_{L^2\to L^2} \le \sup_{\lambda\in[0,\tau]} |e^{2t^2\lambda^2}G(\lambda)| \le e^{2t^2\tau^2}\|G\|_\infty.$$

On the other hand, by (1.3) it readily follows that for $x \in \widetilde{X}$,

$$\|e^{-t^2L}(x,\cdot)\|_{L^2}^2 \le \frac{c}{|B(x,t)|} \int_X \frac{d\mu(y)}{|B(y,t)|(1+t^{-1}\rho(x,y))^{2d+1}} \le c|B(x,t)|^{-1},$$

where we have used (2.1)–(2.2).

Now, applying [2, Proposition 2.9] we conclude that $G(\sqrt{L})$ is an integral operator with kernel $G(\sqrt{L})(x,y)$ satisfying

$$\begin{aligned} |G(\sqrt{L})(x,y)| &\le \|e^{-t^2L}(x,\cdot)\|_2 \|e^{2t^2L}G(\sqrt{L})\|_{2\to2} \|e^{-t^2L}(\cdot,y)\|_2 \\ &\le ce^{2t^2\tau^2}\|G\|_\infty |B(x,t)|^{-1/2}|B(y,t)|^{-1/2} \end{aligned}$$

for all $x,y \in \widetilde{X}$ and $t > 0$. Therefore, choosing $t = \tau^{-1}$ we arrive at (2.6). ∎

Just as in [7, proof of Theorem 3.4], Propositions 2.1 and 2.2 yield the following important localization result:

THEOREM 2.3. *Suppose $G \in C^m(\mathbb{R})$, $m \ge d+1$, G is real-valued and even, and*

$$|G^{(\nu)}(\lambda)| \le A_m(1+\lambda)^{-r} \quad \textit{for } \lambda \ge 0 \textit{ and } 0 \le \nu \le m, \textit{ where } r > m+d.$$

Then $G(t\sqrt{L})$ is an integral operator with kernel $G(t\sqrt{L})(x,y)$ satisfying

$$|G(t\sqrt{L})(x,y)| \le cA_m|B(x,t)|^{-1}(1+t^{-1}\rho(x,y))^{-m+d/2}, \ \forall t>0, \forall x,y \in \widetilde{X},$$

where $c > 0$ is a constant depending only on r, m and the structural constants $c_0, C^\star, c^\star$.

The action of an operator on the kernel of another operator is clarified by

LEMMA 2.4. *Let the functions F and G satisfy the hypotheses of Theorem 2.3 with $m \ge 3d/2 + 1$. Let H be a real-valued measurable function on $\mathbb{R}_+$ such that*

$$F(\lambda) = H(\lambda)G(\lambda) \quad \text{for almost all } \lambda \in \mathbb{R}_+. \tag{2.7}$$

Then $F(\sqrt{L})$ and $G(\sqrt{L})$ are self-adjoint bounded operators on L^2, and $H(\sqrt{L})$ is a self-adjoint operator (defined densely in X) such that for almost all $x \in X$, $G(\sqrt{L})(x,\cdot) \in D(H(\sqrt{L}))$, and for almost all $x \in X$,

$$F(\sqrt{L})(x,y) = H(\sqrt{L})[G(\sqrt{L})(x,\cdot)](y) \quad \text{for a.a. } y \in X. \tag{2.8}$$

Above, $D(H(\sqrt{L}))$ stands for the domain of $H(\sqrt{L})$, and $F(\sqrt{L})(x,y)$ and $G(\sqrt{L})(x,y)$ are the kernels of the operators $F(\sqrt{L})$ and $G(\sqrt{L})$.

Proof. We shall use the abbreviated notation $F := F(\sqrt{L})$, $G := G(\sqrt{L})$, and $H := H(\sqrt{L})$, and $F(x,y) := F(\sqrt{L})(x,y)$ and $G(x,y) := G(\sqrt{L})(x,y)$ for the kernels of the operators F and G.

Observe that from that fact that the functions F, G, and H are real-valued and measurable it follows that (see e.g. [9]) the operators F, G, and H are self-adjoint. As F and G satisfy the hypotheses of Theorem 2.3, where $m \ge 3d/2 + 1$, we have $F(x,y) = F(y,x)$ and $G(x,y) = G(y,x)$ for a.a. $x, y \in X$, and using (2.2) we get

$$\sup_{x\in X} \int_X |F(x,y)|\, d\mu(y) < \infty, \qquad \sup_{x\in X} \int_X |G(x,y)|\, d\mu(y) < \infty.$$

Hence, the operators F and G are bounded on $L^2(X)$. Moreover, by Theorem 2.3 and (2.2) it follows that

$$\|F(x,\cdot)\|_2^2 = \int_X |F(x,y)|^2\, d\mu(y) \le c|B(x,1)|^{-1}, \qquad \forall x \in \widetilde{X}. \tag{2.9}$$

We claim that

$$G(x,\cdot) \in D(H^*) = D(H) \quad \text{for a.a. } x \in X. \tag{2.10}$$

To prove this we first observe that as is well known [9], $f \in D(H^*)$ if

$$\Big|\int_X (Hg)(y)\overline{f(y)}\, d\mu(y)\Big| \le c\|g\|_2, \qquad \forall g \in \tilde{D}(H),$$

for some constant $c > 0$, where $\tilde{D}(H)$ is a dense subspace of $D(H)$. By (2.7) it follows that $Fg = (GH)g$ for all $g \in D(H)$, and hence for every $g \in D(H)$,

$$\int_X F(x,y)g(y)\, d\mu(y) = \int_X G(x,y)(Hg)(y)\, d\mu(y) \quad \text{for a.a. } x \in X. \tag{2.11}$$

By Assumption I it readily follows that for any fixed $x_0 \in X$,

$$L^2(X) = \overline{\bigcup_{n\geq 1} L^2(\overline{B(x_0,n)})}$$

and $\overline{B(x_0,n)}$ is compact. Hence $L^2(\overline{B(x_0,n)})$ is separable, implying that $L^2(X)$ is separable.

From the fact that $D(H)$ is dense in $L^2(X)$ it follows that there exists an orthonormal basis $\{\varphi_j\}_{j\geq 1}$ for $L^2(X)$ such that $\{\varphi_j\}_{j\geq 1} \subset D(H)$. Indeed, let $\{f_j\}_{j\geq 1} \subset L^2(X)$ be dense in $L^2(X)$. The fact that $D(H)$ is dense in $L^2(X)$ implies that for any $j \in \mathbb{N}$ there exists a sequence $\{g_{jk}\}_{k\geq 1} \subset D(H)$ such that $\|f_j - g_{jk}\|_{L^2} < 1/k$ and hence the countable set $\{g_{jk}\}_{j,k\geq 1} \subset D(H)$ is dense in $L^2(X)$. Removing linearly dependent elements from $\{g_{jk}\}_{j,k\geq 1}$ and applying Gram–Schmidt orthogonalization leads to the existence of the claimed orthonormal basis.

Now, by (2.11) for each $j \in \mathbb{N}$ there exists a set $X_j \subset X$ such that $\mu(X \setminus X_j) = 0$ and

$$\int_X F(x,y)\varphi_j(y)\,d\mu(y) = \int_X G(x,y)(H\varphi_j)(y)\,d\mu(y), \qquad \forall x \in X_j. \tag{2.12}$$

Let $\tilde{D}(H)$ be the linear subspace of $D(H)$ consisting of all finite linear combinations of elements from $\{\varphi_j\}_{j\geq 1}$ and write $X_0 := \bigcap_{j\geq 1} X_j \cap \widetilde{X}$. Clearly $\mu(X \setminus X_0) = 0$. By (2.12),

$$\int_X F(x,y)g(y)\,d\mu(y) = \int_X G(x,y)(Hg)(y)\,d\mu(y), \qquad \forall x \in X_0,\ \forall g \in \tilde{D}(H).$$

From this and (2.9) we get, for all $x \in X_0$ and $g \in \tilde{D}(H)$,

$$\Big|\int_X (Hg)(y)G(x,y)\,d\mu(y)\Big| \leq \|F(x,\cdot)\|_2\|g\|_2 \leq c|B(x,1)|^{-1/2}\|g\|_2.$$

Since $\tilde{D}(H)$ is a dense subspace of $L^2(X)$, the above implies the validity of (2.10).

Using the self-adjointness of H, (2.10), and the fact that $G(x,y)$ is real-valued we obtain, for every $f \in D(H)$ and a.a. $x \in X$,

$$\begin{aligned}(GH)f(x) &= \int_X G(x,y)Hf(y)\,d\mu(y) = \int_X Hf(y)G(x,y)\,d\mu(y)\\ &= \int_X f(y)\overline{H^*(\overline{G(x,\cdot)})}(y)\,d\mu(y) = \int_X H[G(x,\cdot)](y)f(y)\,d\mu(y).\end{aligned}$$

This and (2.11) imply $F(x,\cdot) = H[G(x,\cdot)](\cdot)$ almost everywhere for almost all $x \in X$, as claimed. ∎

We shall frequently use the following basic convergence results.

Proposition 2.5. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be real-valued and even, and $\varphi(0) = 1$. Then for every $f \in L^2(X)$,*

$$f = \lim_{t\to 0} \varphi(t\sqrt{L})f \qquad (\textit{convergence in } L^2). \tag{2.13}$$

Furthermore, if $f, f_j \in L^2(X)$, $j = 1, 2, \ldots$, and $f_j \to f$ in L^2, then for any $t > 0$,

$$\varphi(t\sqrt{L})f(x) = \lim_{j\to\infty} \varphi(t\sqrt{L})f_j(x), \qquad \forall x \in \widetilde{X}. \tag{2.14}$$

Proof. Identity (2.13) is immediate from spectral L^2-theory [13].

To prove (2.14) we note that from Theorem 2.3 it follows that $\varphi(t\sqrt{L})$ is an integral operator with kernel $\varphi(t\sqrt{L})(x, y)$ such that for any $\sigma > 0$ there exists a constant $c_\sigma > 0$ such that

$$|\varphi(t\sqrt{L})(x, y)| \le c_\sigma |B(x, t)|^{-1}(1 + t^{-1}\rho(x, y))^\sigma, \qquad \forall x, y \in \widetilde{X}. \tag{2.15}$$

Identity (2.14) is immediate from (2.15) and (2.2). ∎

3. Hardy spaces via maximal operators. In this section we introduce several maximal operators and establish the equivalence of the norm $\|f\|_{H^p}$ on the maximal Hardy spaces H^p, $0 < p \le 1$, with the respective norms defined by maximal operators. As in the classical case on $\mathbb{R}^n$, the grand maximal operator will play an important rôle.

3.1. Maximal operators and definition of H^p

Definition 3.1. A function $\varphi \in \mathcal{S}(\mathbb{R})$ is called *admissible* if φ is real-valued and even. We introduce the following norms on admissible functions in $\mathcal{S}(\mathbb{R})$:

$$\mathcal{N}_N(\varphi) := \sup_{u\ge 0} (1 + u)^N \max_{0\le m\le N} |\varphi^{(m)}(u)|, \qquad N \ge 0. \tag{3.1}$$

Observe that in the above we only need the values $\varphi(u)$ for $u \ge 0$. Therefore, the condition "φ is even" can be replaced by $\varphi^{(2\nu+1)}(0) = 0$ for $\nu = 0, 1, \ldots$, which implies that the even extension of φ from $\mathbb{R}_+$ to $\mathbb{R}$ will have the required properties.

Definition 3.2. Let φ be an admissible function in $\mathcal{S}(\mathbb{R})$. For any function $f \in L^2(X)$ we define

$$M(f;\varphi)(x) := \sup_{t>0} |\varphi(t\sqrt{L})f(x)|, \qquad \forall x \in \widetilde{X},$$

$$M_a^*(f;\varphi)(x) := \sup_{t>0} \sup_{y\in\widetilde{X},\, \rho(x,y)\le at} |\varphi(t\sqrt{L})f(y)|, \qquad \forall x \in X,\ a \ge 1,$$

$$M_\gamma^{**}(f;\varphi)(x) := \sup_{t>0} \sup_{y\in\widetilde{X}} \frac{|\varphi(t\sqrt{L})f(y)|}{(1 + \rho(x, y)/t)^\gamma}, \qquad \forall x \in X,\ \gamma > 0.$$

Observe that for any $f \in L^2(X)$,

$$M(f;\varphi)(x) \le M_a^*(f;\varphi)(x), \qquad \forall x \in \widetilde{X}, \tag{3.2}$$

$$M_a^*(f;\varphi)(x) \le (1+a)^\gamma M_\gamma^{**}(f;\varphi)(x), \quad \forall x \in X. \tag{3.3}$$

We now introduce the grand maximal operator.

Definition 3.3. Denote

$$\mathcal{F}_N := \{\varphi \in \mathcal{S}(\mathbb{R}) : \varphi \text{ is admissible and } \mathcal{N}_N(\varphi) \le 1\}.$$

The *grand maximal operator* is defined by

$$\mathcal{M}_N(f)(x) := \sup_{\varphi \in \mathcal{F}_N} M_1^*(f;\varphi)(x), \quad \forall x \in X,\ f \in L^2(X),$$

that is,

$$\mathcal{M}_N(f)(x) := \sup_{\varphi \in \mathcal{F}_N} \sup_{t>0} \sup_{y \in \widetilde{X},\, \rho(x,y) \le t} |\varphi(t\sqrt{L})f(y)|, \tag{3.4}$$

where $N > 0$ is sufficiently large (to be specified).

It is readily seen that for any admissible function φ and $a \ge 1$ one has

$$M_a^*(f;\varphi) \le a^N \mathcal{N}_N(\varphi)\mathcal{M}_N(f), \quad \forall f \in L^2(X). \tag{3.5}$$

We shall also use the following version of the Hardy–Littlewood maximal operator:

$$M_\theta f(x) := \sup_{B \ni x} \left(\frac{1}{|B|} \int_B |f(y)|^\theta \, d\mu(y) \right)^{1/\theta}, \quad \theta > 0. \tag{3.6}$$

Now, the maximal inequality takes the form (see e.g. [12]): if $0 < \theta < p$, then

$$\|M_\theta f\|_{L^p} \le c\|f\|_{L^p}, \quad \forall f \in L^p(X). \tag{3.7}$$

In the following we exhibit some important relations between the maximal operators. We begin with a simple estimate showing that $M_\gamma^{**}(f;\varphi)(x)$ is finite almost everywhere for $f \in L^2$.

Proposition 3.4. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be admissible and $\gamma > 2d$. Then for any $f \in L^2(X)$,*

$$M_\gamma^{**}(f;\varphi)(x) \le cM_1(f)(x), \quad \forall x \in X, \tag{3.8}$$

where M_1 is from (3.6), *and hence $M_\gamma^{**}(f;\varphi)(x) < \infty$ for almost all $x \in X$.*

Proof. By Theorem 2.3 it follows that $\varphi(t\sqrt{L})$ is an integral operator with kernel $\varphi(t\sqrt{L})(x,y)$ obeying

$$|\varphi(t\sqrt{L})(x,y)| \le c_\sigma |B(x,t)|^{-1}(1+t^{-1}\rho(x,y))^{-\sigma}, \quad \forall t>0,\ \forall x,y \in \widetilde{X},$$

for an arbitrary $\sigma > 0$. We choose $\sigma := \gamma$. Then from the above, for $x \in X$ and $y \in \widetilde{X}$, we have

$$\frac{|\varphi(t\sqrt{L})f(y)|}{(1+\rho(x,y)/t)^{\gamma}} \le c\int_X \frac{|f(z)|\,d\mu(z)}{|B(z,t)|(1+t^{-1}\rho(z,y))^{\gamma}(1+t^{-1}\rho(x,y))^{\gamma}}$$
$$\le c\int_X \frac{|f(z)|\,d\mu(z)}{|B(z,t)|(1+t^{-1}\rho(z,x))^{\gamma}}$$
$$\le \frac{c}{|B(x,t)|}\int_X \frac{|f(z)|\,d\mu(z)}{(1+t^{-1}\rho(x,z))^{\gamma-d}},$$

using the inequality $(1+t^{-1}\rho(z,y))(1+t^{-1}\rho(x,y)) \ge 1+t^{-1}\rho(z,x)$ and (2.1). Further, we have

$$\int_X \frac{|f(z)|\,d\mu(z)}{(1+t^{-1}\rho(x,z))^{\gamma-d}} = \int_{B(x,t)} \frac{|f(z)|\,d\mu(z)}{(1+t^{-1}\rho(x,z))^{\gamma-d}} + \sum_{m=1}^{\infty}\int_{B(x,t2^m)\setminus B(x,t2^{m-1})} \frac{|f(z)|\,d\mu(z)}{(1+t^{-1}\rho(x,z))^{\gamma-d}}$$
$$\le c\sum_{m=0}^{\infty}\frac{|B(x,t2^m)|}{2^{m(\gamma-d)}}\,\frac{1}{|B(x,t2^m)|}\int_{B(x,t2^m)}|f(z)|\,d\mu(z)$$
$$\le cM_1(f)(x)|B(x,t)|\sum_{m=0}^{\infty}\frac{2^{md}}{2^{m(\gamma-d)}} \le cM_1(f)(x)|B(x,t)|.$$

Here we have used (1.2) and $\gamma > 2d$. Putting all of the above together we obtain $|\varphi(t\sqrt{L})f(y)|(1+\rho(x,y)/t)^{-\gamma} \le cM_1(f)(x)$, which implies (3.8).

In turn by (3.8) and the maximal inequality it follows that

$$\|M_{\gamma}^{**}(f;\varphi)\|_2 \le c\|M_1(f)\|_2 \le c\|f\|_2 < \infty,$$

implying $M_{\gamma}^{**}(f;\varphi)(x) < \infty$ for almost all $x \in X$. ∎

Proposition 3.5. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be admissible and $\varphi(0) \ne 0$. Assume $f \in L^2(X)$.*

(a) *If $0 < \theta \le 1$ and $\gamma > 2d/\theta$, then*

$$M_{\gamma}^{**}(f;\varphi)(x) \le cM_{\theta}(M(f;\varphi))(x) \quad \text{for a.a. } x \in X, \tag{3.9}$$

where $c = c(\theta,\gamma,d,\varphi)$.

(b) *If $0 < \theta \le 1$ and $N > 6d/\theta + 3d/2 + 2$, then*

$$\mathcal{M}_N(f)(x) \le cM_{\theta}(M(f;\varphi))(x) \quad \text{for a.a. } x \in X, \tag{3.10}$$

where $c = c(\theta,d,\varphi)$.

For the proof of this proposition (and later) we shall need

LEMMA 3.6. *Suppose $\varphi \in \mathcal{S}(\mathbb{R})$ is admissible and $\varphi(0) = 1$, and let $N \geq 0$. Then there exist even real-valued functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R})$ with $\psi_0(0) = 1$, $\psi^{(\nu)}(0) = 0$ for $\nu = 0, 1, \ldots, N$, and such that for any $f \in L^2(X)$ and $j \in \mathbb{Z}$,*

$$f = \psi_0(2^{-j}\sqrt{L})\varphi(2^{-j}\sqrt{L})f + \sum_{k=j}^{\infty} \psi(2^{-k}\sqrt{L})[\varphi(2^{-k}\sqrt{L}) - \varphi(2^{-k+1}\sqrt{L})]f, \tag{3.11}$$

where the convergence is in L^2.

Furthermore, under the above conditions on φ, if $\operatorname{supp}\hat{\varphi} \subset [-1, 1]$, then the functions ψ_0 and ψ can be selected so that

$$\begin{aligned} &\operatorname{supp}\widehat{\psi_0} \subset [-2N, 2N], \\ &\operatorname{supp}(\lambda^{-k}\psi(\lambda))^\wedge \subset [-2N, 2N], \qquad k = 0, 1, \ldots, N. \end{aligned} \tag{3.12}$$

Proof. We borrow the idea of this proof from [10, Theorem 1.6]. Evidently,

$$\varphi(\lambda)^2 + \sum_{k=1}^{\infty} [\varphi(2^{-k}\lambda)^2 - \varphi(2^{-k+1}\lambda)^2] = 1, \qquad \lambda \in \mathbb{R},$$

and as $\varphi \in \mathcal{S}(\mathbb{R})$ the series converges absolutely. From the above,

$$1 = \Big(\varphi(\lambda)^2 + \sum_{k=1}^{\infty} [\varphi(2^{-k}\lambda)^2 - \varphi(2^{-k+1}\lambda)^2]\Big)^N.$$

It is easy to see that for $N \geq 1$ this identity can be written in the form

$$\begin{aligned} 1 = &\sum_{m=1}^{N} \binom{N}{m} \varphi(\lambda)^{2m}(1 - \varphi(\lambda)^2)^{N-m} \\ &+ \sum_{k=1}^{\infty}\sum_{m=1}^{N} \binom{N}{m} [\varphi(2^{-k}\lambda)^2 - \varphi(2^{-k+1}\lambda)^2]^m (1 - \varphi(2^{-k}\lambda)^2)^{N-m}, \end{aligned}$$

which leads to

$$\psi_0(\lambda)\varphi(\lambda) + \sum_{k=1}^{\infty} \psi(2^{-k}\lambda)[\varphi(2^{-k}\lambda) - \varphi(2^{-k+1}\lambda)] = 1 \tag{3.13}$$

with

$$\psi_0(\lambda) := \sum_{m=1}^{N} \binom{N}{m} \varphi(\lambda)^{2m-1}(1 - \varphi(\lambda)^2)^{N-m}$$

and

$$\psi(\lambda) := [\varphi(\lambda) + \varphi(2\lambda)] \sum_{m=1}^{N} \binom{N}{m} [\varphi(\lambda)^2 - \varphi(2\lambda)^2]^{m-1}(1 - \varphi(\lambda)^2)^{N-m}. \tag{3.14}$$

Clearly, $\psi_0, \psi \in \mathcal{S}(\mathbb{R})$, ψ_0, ψ are even, $\psi^{(\nu)}(0) = 0$ for $\nu = 0, 1, \ldots, N-2$, and $\psi_0(0) = 1$.

Identity (3.11) follows immediately by (3.13) and spectral theory (see e.g. [13]).

Finally, by replacing N with $N+2$ in the above proof we get what we need.

We now prove the right-hand inclusion in (3.12) for $k = N-1$. The proof of the left-hand inclusion is easier and will be omitted. Using (3.14) we can write

$$\lambda^{-N+1}\psi(\lambda) = [\varphi(\lambda) + \varphi(2\lambda)] \times \sum_{m=1}^{N} \binom{N}{m} \left[\frac{\varphi(\lambda) - \varphi(2\lambda)}{\lambda}(\varphi(\lambda) + \varphi(2\lambda))\right]^{m-1} \left[\frac{1-\varphi(\lambda)}{\lambda}(1+\varphi(\lambda))\right]^{N-m}. \tag{3.15}$$

Clearly, $\operatorname{supp}\hat{\varphi} \subset [-1,1]$ implies $\operatorname{supp}\widehat{\varphi(2\lambda)} \subset [-2,2]$. We next show that

$$\operatorname{supp}\left(\frac{1-\varphi(\lambda)}{\lambda}\right)^{\wedge} \subset [-1,1]. \tag{3.16}$$

By Taylor's theorem and the fact that $\varphi(0) = 1$ we get

$$\frac{\varphi(\lambda) - 1}{\lambda} = \varphi'(0) + \lambda \int_0^1 (1-u)\varphi''(\lambda u)\, du.$$

For the Fourier transform of the above integral we have

$$\begin{aligned}\Big(\int_0^1 (1-u)\varphi''(\lambda u)\, du\Big)^{\wedge}(\xi) &= \int_{\mathbb{R}}\int_0^1 (1-u)\varphi''(\lambda u)\, du\, e^{-i\lambda\xi}\, d\lambda \\ &= \int_0^1 (1-u)\int_{\mathbb{R}} \varphi''(\lambda u) e^{-i\lambda\xi}\, d\lambda\, du = \int_0^1 (1-u)\widehat{\varphi''}(\xi/u)\,\frac{du}{u} \\ &= \int_1^\infty (1-1/v)\widehat{\varphi''}(v\xi)\,\frac{dv}{v}.\end{aligned}$$

The above manipulations are easy to justify since $\varphi \in \mathcal{S}(\mathbb{R})$. Due to the fact that $\operatorname{supp}\hat{\varphi} \subset [-1,1]$ we have $\operatorname{supp}\widehat{\varphi''} \subset [-1,1]$, and from the above it follows that $\operatorname{supp}(\int_0^1 (1-u)\varphi''(\lambda u)\, du)^{\wedge} \subset [-1,1]$. This implies (3.16). From (3.16) it follows that

$$\operatorname{supp}\left(\frac{\varphi(\lambda) - \varphi(2\lambda)}{\lambda}\right)^{\wedge} \subset [-2,2].$$

Clearly, the Fourier transform of $\lambda^{-N+1}\psi(\lambda)$ is represented in terms of the convolutions of the Fourier transforms of all terms in its representation

(3.15), leading to the conclusion that

$$\operatorname{supp}(\lambda^{-N+1}\psi(\lambda))^\wedge \subset [-2N+2, 2N-2].$$

By increasing N as above we arrive at (3.12). ■

Proof of Proposition 3.5. (a) We borrow the idea of this proof from [8, Lemma 3.2]. Assume $0<\theta\le 1$ and $\gamma>2d/\theta$, and let $f\in L^2(X)$. We may assume that $\varphi(0)=1$ for otherwise we use $\varphi(0)^{-1}\varphi$ instead. By Lemma 3.6 there exist even real-valued functions $\psi_0,\psi\in\mathcal{S}(\mathbb{R})$ such that $\psi_0(0)=1$, $\psi^{(\nu)}(0)=0$ for $\nu=0,1,\dots,N$, and (3.11) holds for all $j\in\mathbb{Z}$.

Fix $t>0$ and let $2^{-j}\le t<2^{-j+1}$. Using (3.11) and (2.14) we get, for $x\in X$ and $y\in\widetilde{X}$,

$$\frac{|\varphi(t\sqrt{L})f(y)|}{(1+\rho(x,y)/t)^\gamma}\le c\frac{|\varphi(t\sqrt{L})\psi_0(2^{-j}\sqrt{L})\varphi(2^{-j}\sqrt{L})f(y)|}{(1+2^j\rho(x,y))^\gamma} + c\sum_{k=j}^{\infty}\frac{|\varphi(t\sqrt{L})\psi(2^{-k}\sqrt{L})[\varphi(2^{-k}\sqrt{L})-\varphi(2^{-k+1}\sqrt{L})]f(y)|}{(1+2^j\rho(x,y))^\gamma}.$$

Let $\omega(\lambda):=\varphi(t2^j\lambda)\psi(2^{-(k-j)}\lambda)$. Then $\omega(2^{-j}\sqrt{L})=\varphi(t\sqrt{L})\psi(2^{-k}\sqrt{L})$.

Now, choose $N>3\gamma+3d/2+2$ and set $m:=\lfloor\gamma+d/2+1\rfloor$. As $\varphi,\psi\in\mathcal{S}(\mathbb{R})$ there exists a constant $c>0$ such that for $\nu=0,1,\dots,N$,

$$|\varphi^{(\nu)}(\lambda)|\le c(1+\lambda)^{-N},\quad |\psi^{(\nu)}(\lambda)|\le c(1+\lambda)^{-N},\quad \lambda>0, \tag{3.17}$$

yielding

$$|\omega^{(\nu)}(\lambda)|\le c(1+\lambda)^{-N},\quad \lambda>0,\ \nu=0,1,\dots,N.$$

From this estimate we obtain, for $\lambda\ge 2^{(k-j)/2}$,

$$|\omega^{(\nu)}(\lambda)|\le c(1+\lambda)^{-m-d-1}2^{-(k-j)(N-m-d-1)/2},$$

and using the fact that $N\ge 3\gamma+3d/2+2+2\varepsilon$ for some $\varepsilon>0$, it follows that for $\nu=0,1,\dots,N$,

$$|\omega^{(\nu)}(\lambda)|\le c2^{-(k-j)(\gamma+\varepsilon)}(1+\lambda)^{-m-d-1},\quad \lambda\ge 2^{(k-j)/2}. \tag{3.18}$$

On the other hand, as $\psi^{(\nu)}(0)=0$ for $\nu=0,1,\dots,N$, we use Taylor's formula and (3.17) to obtain $|\psi^{(\nu)}(\lambda)|\le c\lambda^{N-\nu}$, $\lambda>0$, $\nu=0,1,\dots,N$. Hence,

$$\left|\left(\frac{d}{d\lambda}\right)^\nu\psi(2^{-(k-j)}\lambda)\right|\le c2^{-(k-j)N}\lambda^{N-\nu}\le c2^{-(k-j)N/2}\quad \text{for } 0\le\lambda\le 2^{(k-j)/2}.$$

From this estimate and (3.17) we get

$$|\omega^{(\nu)}(\lambda)|\le c2^{-(k-j)N/2}(1+\lambda)^{-N},\quad 0\le\lambda\le 2^{(k-j)/2},\ \nu=0,1,\dots,N.$$

In turn, this estimate and (3.18) imply that (3.18) holds for $0 < \lambda < \infty$. Now, Theorem 2.3 applied to $\omega(2^{-j}\sqrt{L})$ leads to the following estimate on the kernel of the operator $\varphi(t\sqrt{L})\psi(2^{-k}\sqrt{L})$ (recall that $2^{-j} \le t < 2^{-j+1}$):

$$(3.19)\qquad |[\varphi(t\sqrt{L})\psi(2^{-k}\sqrt{L})](x,y)| \le \frac{c2^{-(k-j)(\gamma+\varepsilon)}}{|B(y,2^{-j})|(1+2^j\rho(x,y))^\gamma}, \qquad \forall x,y \in \widetilde{X}.$$

Consequently,

$$\begin{aligned}
&\frac{|\varphi(t\sqrt{L})\psi(2^{-k}\sqrt{L})[\varphi(2^{-k}\sqrt{L})-\varphi(2^{-k+1}\sqrt{L})]f(y)|}{(1+2^j\rho(x,y))^\gamma} \\
&\quad\le c\int_X \frac{2^{-(\gamma+\varepsilon)(k-j)}[|\varphi(2^{-k}\sqrt{L})f(z)|+|\varphi(2^{-k+1}\sqrt{L})f(z)|]\,d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(y,z))^\gamma(1+2^j\rho(x,y))^\gamma} \\
&\quad\le c\int_X \frac{2^{-(\gamma+\varepsilon)(k-j)}[|\varphi(2^{-k}\sqrt{L})f(z)|+|\varphi(2^{-k+1}\sqrt{L})f(z)|]\,d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(x,z))^\gamma} \\
&\quad\le c2^{-(k-j)\varepsilon}\int_X \frac{[|\varphi(2^{-k}\sqrt{L})f(z)|+|\varphi(2^{-k+1}\sqrt{L})f(z)|]\,d\mu(z)}{|B(z,2^{-j})|(1+2^k\rho(x,z))^\gamma} \\
&\quad\le c2^{-(k-j)\varepsilon}[M_\gamma^{**}(f;\varphi)(x)]^{1-\theta} \\
&\qquad\times\int_X \frac{[|\varphi(2^{-k}\sqrt{L})f(z)|^\theta+|\varphi(2^{-k+1}\sqrt{L})f(z)|^\theta]\,d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta}}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\frac{|\varphi(t\sqrt{L})\psi_0(2^{-j}\sqrt{L})\varphi(2^{-j}\sqrt{L})f(y)|}{(1+2^j\rho(x,y))^\gamma} \\
&\qquad\le c[M_\gamma^{**}(f;\varphi)(x)]^{1-\theta}\int_X \frac{|\varphi(2^{-j}\sqrt{L})f(z)|^\theta\,d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta}}.
\end{aligned}$$

Putting the above estimates together we get

$$\begin{aligned}
&\frac{|\varphi(t\sqrt{L})f(y)|}{(1+\rho(x,y)/t)^\gamma} \\
&\qquad\le c[M_\gamma^{**}(f;\varphi)(x)]^{1-\theta}\sum_{k=j}^\infty 2^{-(k-j)\varepsilon}\int_X \frac{|\varphi(2^{-k}\sqrt{L})f(z)|^\theta\,d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta}}.
\end{aligned}$$

At this point we use the fact that $M_\gamma^{**}(f;\varphi)(x) < \infty$ for almost all $x \in X$, established in Proposition 3.4, to conclude that for almost all $x \in X$,

$$[M_\gamma^{**}(f;\varphi)(x)]^\theta \le c\sum_{k=j}^\infty 2^{-(k-j)\varepsilon}\int_X \frac{|\varphi(2^{-k}\sqrt{L})f(z)|^\theta\,d\mu(z)}{|B(x,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta-d}}.$$

Here we have used (2.1) as well. Denote briefly $F(z) := \varphi(2^{-k}\sqrt{L})f(z)$. Just as in the second half of the proof of Proposition 3.4, using $\gamma > 2d/\theta$ we obtain

$$\int_X \frac{|F(z)|^\theta \, d\mu(z)}{|B(x,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta-d}} \le cM_\theta(F)(x)^\theta.$$

Therefore, for almost all $x \in X$,

$$[M_\gamma^{**}(f;\varphi)(x)]^\theta \le c\sum_{k=j}^\infty 2^{-(k-j)\varepsilon} M_\theta(\varphi(2^{-k}\sqrt{L})f)(x)^\theta \le cM_\theta(M(f;\varphi))(x)^\theta,$$

which yields (3.9).

(b) We shall proceed much as in the proof of (a). Let $\phi \in \mathcal{F}_N$ and assume that $\varphi \in \mathcal{S}(\mathbb{R})$ is admissible. Choose $\gamma > 2d/\theta$ so that $N > 3\gamma + 3d/2 + 2$. Then there exists $\varepsilon > 0$ such that $N \ge 3\gamma + 3d/2 + 2 + 2\varepsilon$.

Assume $t > 0$ and let $2^{-j} \le t < 2^{-j+1}$. Just as in the proof of (a), by Lemma 3.6 there exist even real-valued functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R})$ such that $\psi^{(\nu)}(0) = 0$ for $\nu = 0, 1, \dots, N$ and (3.11) holds for any $j \in \mathbb{Z}$. Hence, using (2.14) we obtain, for $f \in L^2(X)$,

$$\begin{aligned}\frac{|\phi(t\sqrt{L})f(y)|}{(1+\rho(x,y)/t)^\gamma} &\le c\frac{|\phi(t\sqrt{L})\psi_0(2^{-j}\sqrt{L})\varphi(2^{-j}\sqrt{L})f(y)|}{(1+2^j\rho(x,y))^\gamma} \\ &\quad + c\sum_{k=j}^\infty \frac{|\phi(t\sqrt{L})\psi(2^{-k}\sqrt{L})[\varphi(2^{-k}\sqrt{L}) - \varphi(2^{-k+1}\sqrt{L})]f(y)|}{(1+2^j\rho(x,y))^\gamma}.\end{aligned}$$

Just as in (3.19) we have

$$|\phi(t\sqrt{L})\psi(2^{-k}\sqrt{L})(x,y)| \le \frac{c2^{-(k-j)(\gamma+\varepsilon)}}{|B(y,2^{-j})|(1+2^j\rho(x,y))^\gamma}, \qquad \forall x, y \in \widetilde{X},$$

where the constant $c > 0$ is independent of ϕ due to $\mathcal{N}_N(\phi) \le 1$. Therefore, as in the proof of (a), for $x \in X$ and $y \in \widetilde{X}$,

$$\begin{aligned}&\frac{|\phi(t\sqrt{L})\psi(2^{-k}\sqrt{L})[\varphi(2^{-k}\sqrt{L}) - \varphi(2^{-k+1}\sqrt{L})]f(y)|}{(1+2^j\rho(x,y))^\gamma} \\ &\quad \le c\int_X \frac{2^{-(\gamma+\varepsilon)(k-j)}[|\varphi(2^{-k}\sqrt{L})f(z)| + |\varphi(2^{-k+1}\sqrt{L})f(z)|]\, d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(y,z))^\gamma(1+2^j\rho(x,y))^\gamma} \\ &\quad \le c2^{-(k-j)\varepsilon}[M_\gamma^{**}(f;\varphi)(x)]^{1-\theta} \\ &\qquad \times \int_X \frac{[|\varphi(2^{-k}\sqrt{L})f(z)|^\theta + |\varphi(2^{-k+1}\sqrt{L})f(z)|^\theta]\, d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta}}.\end{aligned}$$

Similarly

$$\frac{|\phi(t\sqrt{L})\psi_0(2^{-j}\sqrt{L})\varphi(2^{-j}\sqrt{L})f(y)|}{(1+2^j\rho(x,y))^\gamma} \le c[M_\gamma^{**}(f;\varphi)(x)]^{1-\theta}\int_X \frac{|\varphi(2^{-j}\sqrt{L})f(z)|^\theta\,d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta}}.$$

Here the constant $c > 0$ is independent of ϕ since $\mathcal{N}_N(\phi) \le 1$. As before, denoting $F(z) := \varphi(2^{-k}\sqrt{L})f(z)$ we obtain

$$\int_X \frac{|F(z)|^\theta\,d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta}} \le c\int_X \frac{|F(z)|^\theta\,d\mu(z)}{|B(x,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta-d}} \le cM_\theta(F)(x)^\theta.$$

Therefore,

$$\begin{aligned}\frac{|\phi(t\sqrt{L})f(y)|}{(1+\rho(x,y)/t)^\gamma} &\le c[M_\gamma^{**}(f;\varphi)(x)]^{1-\theta}\\ &\quad\times\sum_{k=j}^\infty 2^{-(k-j)\varepsilon}\int_X \frac{|\varphi(2^{-k}\sqrt{L})f(z)|^\theta\,d\mu(z)}{|B(z,2^{-j})|(1+2^j\rho(x,z))^{\gamma\theta}}\\ &\le c[M_\gamma^{**}(f;\varphi)(x)]^{1-\theta}\sum_{k=j}^\infty 2^{-(k-j)\varepsilon}[M_\theta(|\varphi(2^{-k}\sqrt{L})f|)(x)]^\theta\\ &\le c[M_\gamma^{**}(f;\varphi)(x)]^{1-\theta}[M_\theta(M(f;\varphi))(x)]^\theta\sum_{k=j}^\infty 2^{-(k-j)\varepsilon}\\ &\le cM_\theta(M(f;\varphi))(x),\end{aligned}$$

where for the last estimate we have used the fact that $M_\gamma^{**}(f;\varphi)(x) \le cM_\theta(M(f;\varphi))(x)$ for almost all $x \in X$, by (3.9). Thus for almost all $x \in X$,

$$\sup_{t>0}\sup_{y\in\widetilde{X},\,\rho(x,y)\le t}|\phi(t\sqrt{L})f(y)| \le 2^\gamma\sup_{t>0}\sup_{y\in\widetilde{X}}\frac{|\phi(t\sqrt{L})f(y)|}{(1+\rho(x,y)/t)^\gamma} \le cM_\theta(M(f;\varphi))(x),$$

which completes the proof. ∎

Proposition 3.5 leads to the following

THEOREM 3.7. *Let $0 < p \le 1$. Then for any $N > 6d/p + 3d/2 + 2$, $\gamma > 2d/p$, $a \ge 1$, and an admissible $\varphi \in \mathcal{S}(\mathbb{R})$ with $\varphi(0) \ne 0$ we have, for all $f \in L^2(X)$,*

$$\|f\|_{H^p} \sim \|\mathcal{M}_N(f)\|_{L^p} \sim \|M(f;\varphi)\|_{L^p} \sim \|M_a^*(f;\varphi)\|_{L^p} \sim \|M_\gamma^{**}(f;\varphi)\|_{L^p}. \tag{3.20}$$

Here the constants in the equivalences involving φ depend not only on the parameters but on φ as well.

Proof. Write $\Phi(\lambda) := e^{-\lambda^2}$. Apparently $\Phi \in \mathcal{S}(\mathbb{R})$, Φ is admissible, and $\Phi(0) \neq 0$. Let $N > 6d/p + 3d/2 + 2$ and choose θ so that $0 < \theta < p$ and $N > 6d/\theta + 3d/2 + 2$. Then applying Proposition 3.5(b) we obtain, for a given $f \in L^2(X)$,

$$\|\mathcal{M}_N(f)\|_p \le c\|M_\theta(M(f;\Phi))\|_p \le c\|M(f;\Phi)\|_p = c\|f\|_{H^p},$$

where we have used the maximal inequality (3.7).

In the other direction, using (3.2) and (3.5) we get

$$\|f\|_{H^p} = \|M(f;\Phi)\|_p \le \|M_1^*(f;\Phi)\|_p \le c\|\mathcal{M}_N(f)\|_p.$$

Thus the first equivalence in (3.20) is established.

Just in the same way we get $\|\mathcal{M}_N(f)\|_{L^p} \sim \|M(f;\varphi)\|_{L^p}$ with constants of equivalence depending in addition on φ. We choose θ so that $0 < \theta < p$ and $\gamma > 2d/\theta$ and apply Proposition 3.5(a) and the maximal inequality as above to obtain $\|M_\gamma^{**}(f;\varphi)\|_{L^p} \le c\|M(f;\varphi)\|_{L^p}$. All other estimates we need follow from (3.3) and (3.5). ■

4. Equivalence of maximal and atomic Hardy spaces. In this section we present the proof of Theorem 1.4. We shall first carry out the proof in the noncompact case and then explain the modifications that need to be made in the compact case.

4.1. Proof of the embedding $H^p \subset H^p_A$ in the noncompact case. We show that if $f \in H^p$, $0 < p \le 1$, then $f \in H^p_A$ and $\|f\|_{H^p_A} \le c\|f\|_{H^p}$.

The decomposition of L^2-functions from Lemma 3.6 will play a central rôle in this proof. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be real-valued and even, $\varphi(0) = 1$, and with Fourier transform $\hat{\varphi}$ obeying $\operatorname{supp}\hat{\varphi} \subset [-1,1]$. Fix an integer $N > 6d/p+3d/2+2$ as in Theorem 3.7. By Lemma 3.6 there exist even real-valued functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R})$ with $\psi_0(0) = 1$, $\psi^{(\nu)}(0) = 0$ for $\nu = 0, 1, \dots, N$,

$$\operatorname{supp}\hat{\psi}_0 \subset [-2N, 2N],\ \operatorname{supp}(\lambda^{-\nu}\psi(\lambda))^\wedge \subset [-2N, 2N],\ \nu = 0, \dots, N, \tag{4.1}$$

such that for any $f \in L^2(X)$ and $j \in \mathbb{Z}$,

$$f = \psi_0(2^{-j}\sqrt{L})\varphi(2^{-j}\sqrt{L})f + \sum_{k=j}^{\infty} \psi(2^{-k}\sqrt{L})[\varphi(2^{-k}\sqrt{L}) - \varphi(2^{-k+1}\sqrt{L})]f, \tag{4.2}$$

where the convergence is in L^2.

Write $\tilde\varphi := \varphi_0$ and $\tilde\psi(\lambda) := \varphi(\lambda) - \varphi(2\lambda)$. From now on we shall use the following more compact notation:

(4.3)

$$\varphi_k := \varphi(2^{-k}\sqrt{L}), \quad \tilde\varphi_k := \psi_0(2^{-k}\sqrt{L}), \quad \psi_k := \psi(2^{-k}\sqrt{L}), \quad \tilde\psi_k := \tilde\psi(2^{-k}\sqrt{L}).$$

Clearly, these are integral operators whose kernels will be denoted by $\varphi_k(x,y)$, $\tilde\varphi_k(x,y)$, $\psi_k(x,y)$, and $\tilde\psi_k(x,y)$. Observe that since φ, $\tilde\varphi$, ψ, and $\tilde\psi$ are real-valued, we have $\varphi_k(y,x) = \varphi_k(x,y)$ for all $x,y \in \widetilde{X}$, and similarly for the others.

Also, $\varphi, \tilde\varphi, \psi, \tilde\psi \in \mathcal{S}(\mathbb{R})$, $\varphi, \tilde\varphi, \psi, \tilde\psi$ are even, and from $\operatorname{supp}\hat\varphi \subset [-1,1]$ and (4.1) it follows by the final speed propagation property (Proposition 2.1) that there exists a constant $\tau > 1$ such that

$$\operatorname{supp}\varphi_k(x,\cdot), \operatorname{supp}\tilde\varphi_k(x,\cdot), \operatorname{supp}\psi_k(x,\cdot), \operatorname{supp}\tilde\psi_k(x,\cdot) \subset B(x,\tau 2^{-k}), \tag{4.4}$$

and

$$\operatorname{supp}[L^{-\nu}\psi(2^{-k}\sqrt{L})](x,\cdot) \subset B(x,\tau 2^{-k}), \qquad \nu = 0,1,\dots,N. \tag{4.5}$$

By Proposition 2.2, (1.2), and (2.1) it follows that for any $x \in X$,

$$|\varphi_k(x,y)|, |\tilde\varphi_k(x,y)|, |\psi_k(x,y)|, |\tilde\psi_k(x,y)| \le c|B(x,2^{-k})|^{-1}, \ \forall y \in \widetilde{X}. \tag{4.6}$$

We shall utilize the following assertion involving the grand maximal operator $\mathcal{M}_N$, defined in (3.4): Let $\phi \in \mathcal{S}(\mathbb{R})$ be admissible and assume $\mathcal{N}_N(\phi) \le c$. Then for any $f \in L^2(X)$, $k \in \mathbb{Z}$, and $x \in X$,

$$|\phi(2^{-k}\sqrt{L})f(y)| \le c\mathcal{M}_N(f)(x) \quad \forall y \in \widetilde{X} \text{ with } \rho(x,y) \le 2\tau 2^{-k}, \tag{4.7}$$

where $\tau > 1$ is the constant from (4.4)–(4.5). This claim follows readily from (3.2) and (3.5).

We next establish an estimate of a similar nature for the operators defined by

$$Q_{\ell j} := \sum_{k=\ell}^{j} \psi_k\tilde\psi_k, \qquad -\infty < \ell \le j < \infty. \tag{4.8}$$

Lemma 4.1. *For any $f \in L^2(X)$ and $x \in X$,*

$$|Q_{\ell j}f(y)| \le c\mathcal{M}_N(f)(x) \quad \text{for all } y \in \widetilde{X} \text{ with } \rho(x,y) \le 2\tau 2^{-j}. \tag{4.9}$$

Furthermore,

$$|Q_{\ell j}f(x)| \le cM_1(f\cdot \mathbb{1}_{B(x,2\tau 2^{-\ell})})(x), \qquad \forall x \in \widetilde{X}, \tag{4.10}$$

where M_1 is the Hardy–Littlewood maximal operator defined in (3.6), *and $c > 0$ is a constant independent of f, ℓ, j.*

Proof. Let $K := j - \ell$ and set $Q(\lambda) := \sum_{k=0}^{K}\psi(2^k\lambda)\tilde\psi(2^k\lambda)$. It is readily seen that $Q_{\ell j} = Q(2^{-j}\sqrt{L})$. Clearly, Q is even and $Q \in \mathcal{S}(\mathbb{R})$. We next show that

$$\mathcal{N}_N(Q) \le c < \infty, \tag{4.11}$$

where the constant $c > 0$ is independent of ℓ, j.

Set $q(\lambda) := \psi(\lambda)\tilde{\psi}(\lambda)$. Evidently, $q \in \mathcal{S}(\mathbb{R})$, q is real-valued and even, and $q^{(\nu)}(0) = 0$, $\nu = 0, 1, \dots, N$. We claim that for any $\sigma > 0$ there exists a constant $c > 0$ depending on σ, N, and φ such that

$$|q^{(\nu)}(\lambda)| \le \frac{c|\lambda|^{N-\nu+1}}{(1+|\lambda|)^{\sigma+N}}, \qquad \lambda \in \mathbb{R},\ \nu = 0, 1, \dots, N. \tag{4.12}$$

Indeed, by Taylor's theorem

$$q^{(\nu)}(\lambda) = \frac{\lambda^{N+1-\nu}}{(N+1-\nu)!}\int_0^1 (1-u)^{N-\nu} q^{(N+1)}(\lambda u)\, du, \qquad 0 \le \nu \le N,$$

and hence

$$|q^{(\nu)}(\lambda)| \le c\|q^{(N+1)}\|_\infty |\lambda|^{N+1-\nu}. \tag{4.13}$$

On the other hand, since $q \in \mathcal{S}(\mathbb{R})$ we have

$$|q^{(\nu)}(\lambda)| \le \frac{c}{(1+|\lambda|)^{\sigma+N}}. \tag{4.14}$$

By considering two cases: $|\lambda| \le 1$ and $|\lambda| \ge 1$, estimates (4.13)–(4.14) imply (4.12). From (4.12) we get

$$|Q^{(\nu)}(\lambda)| \le c\sum_{k=0}^{\infty} \frac{|2^k\lambda|^{N-\nu+1}}{(1+|2^k\lambda|)^{\sigma+N}}.$$

To estimate the above sum we choose $\sigma = N$ and consider two cases.

Case 1: $|\lambda| \le 1$. Assume $2^{-m-1} < |\lambda| \le 2^{-m}$. Then from the above,

$$|Q^{(\nu)}(\lambda)| \le c\sum_{k=0}^{\infty} \frac{2^{(k-m)(N-\nu+1)}}{(1+2^{k-m})^{\sigma+N}} \le c\sum_{k=0}^{m} 2^{(k-m)(N-\nu+1)} + c\sum_{k=m+1}^{\infty} 2^{-(k-m)\sigma},$$

implying $|Q^{(\nu)}(\lambda)| \le c \le c'(1+|\lambda|)^{-N}$.

Case 2: $|\lambda| > 1$. We have

$$|Q^{(\nu)}(\lambda)| \le c\sum_{k=0}^{\infty} \frac{1}{(1+|2^k\lambda|)^N} \le \frac{c}{(1+|\lambda|)^N}\sum_{k=0}^{\infty}\frac{1}{2^{kN}} \le \frac{c}{(1+|\lambda|)^N}.$$

In both cases we have

$$|Q^{(\nu)}(\lambda)| \le c(1+|\lambda|)^{-N} \qquad \text{for } \nu = 0, 1, \dots, N, \tag{4.15}$$

which confirms (4.11) (see (3.1)). In turn, just as in (4.7), estimate (4.11) yields

$$|Q(2^{-j}\sqrt{L})f(y)| \le c\mathcal{M}_N(f)(x) \quad \text{for all}\, y \in \widetilde{X} \text{ with } \rho(x,y) \le 2\tau 2^{-j},$$

which verifies (4.9).

To prove (4.10) we note that by (4.15) and Theorem 2.3 it follows that $Q_{\ell j}$ is an integral operator with kernel $Q_{\ell j}(x,y)$ satisfying

$$|Q_{\ell j}(x,y)| \le c|B(x,2^{-j})|^{-1}(1+2^j\rho(x,y))^{-N+3d/2+1}, \quad \forall x,y \in \widetilde{X}.$$

Now, just as in the proof of Proposition 3.4, this implies (4.10), on taking into account that $N - 3d/2 - 1 > d$ and (4.4). ∎

From (4.2) it follows that for any $f \in L^2(X)$,

$$f = \varphi_j \tilde{\varphi}_j f + \sum_{k \ge j} \psi_k \tilde{\psi}_k f \quad (\text{convergence in } L^2), \tag{4.16}$$

which readily implies

$$f = \sum_{k \in \mathbb{Z}} \psi_k \tilde{\psi}_k f \quad (\text{convergence in } L^2). \tag{4.17}$$

Let $f \in H^p \cap L^2$, $0 < p \le 1$, and $f \ne 0$. We define

$$\Omega_r := \{x \in X : \mathcal{M}_N(f)(x) > 2^r\}, \quad r \in \mathbb{Z}. \tag{4.18}$$

Clearly, $\Omega_{r+1} \subset \Omega_r$ and $X = \bigcup_{r\in\mathbb{Z}} \Omega_r$. The latter identity follows by the inequality $\mathcal{M}_N(f)(x) > 0$ for all $x \in X$ due to $f \ne 0$. Also, Ω_r is open because $\mathcal{M}_N(f)(x)$ is lower semicontinuous. From the definition of Ω_r in (4.18) it readily follows that

$$\begin{aligned}\sum_{r\in\mathbb{Z}} 2^{pr}|\Omega_r| &= \sum_{r\in\mathbb{Z}} 2^{pr} \sum_{\nu \ge r} |\Omega_\nu \setminus \Omega_{\nu+1}| = \sum_{\nu\in\mathbb{Z}} |\Omega_\nu \setminus \Omega_{\nu+1}| \sum_{r \le \nu} 2^{pr} \\ &\le c_p \sum_{\nu\in\mathbb{Z}} 2^{p\nu} |\Omega_\nu \setminus \Omega_{\nu+1}| \le c_p \sum_{\nu\in\mathbb{Z}} \int_{\Omega_\nu\setminus\Omega_{\nu+1}} \mathcal{M}_N(f)(x)^p \, d\mu(x) \\ &= c_p \int_X \mathcal{M}_N(f)(x)^p \, d\mu(x).\end{aligned}$$

Hence,

$$\sum_{r\in\mathbb{Z}} 2^{pr}|\Omega_r| \le c \int_X \mathcal{M}_N(f)(x)^p \, d\mu(x) \le c\|f\|_{H^p}^p, \tag{4.19}$$

implying

$$|\Omega_r| \le c2^{-pr}\|f\|_{H^p}^p, \quad r \in \mathbb{Z}. \tag{4.20}$$

Assume $\Omega_r \ne \emptyset$ and write

(4.21)

$$E_{rk} := \{x \in \Omega_r : \operatorname{dist}(x, \Omega_r^c) > 2\tau 2^{-k}\} \setminus \{x \in \Omega_{r+1} : \operatorname{dist}(x, \Omega_{r+1}^c) > 2\tau 2^{-k}\}.$$

We define

$$F_r := \sum_{k\in\mathbb{Z}} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y), \qquad r\in\mathbb{Z}, \tag{4.22}$$

and in general

$$F_{r,\kappa_0,\kappa_1} := \sum_{k=\kappa_0}^{\kappa_1} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y), \qquad -\infty\le\kappa_0\le\kappa_1\le\infty, \tag{4.23}$$

where the convergence is in L^2. As will be shown in Lemma 4.2 below, the functions F_r and F_{r,κ_0,κ_1} are well defined and $F_r, F_{r,\kappa_0,\kappa_1} \in L^2\cap L^\infty$.

Clearly, for every $x\in\Omega_r$ there exists $s_{rx}\in\mathbb{Z}$ such that $E_{rk}\cap B(x,\tau 2^{-k}) = \emptyset$ for $k<s_{rx}$. This coupled with (4.4) implies

$$\int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = 0, \qquad \forall k<s_{rx},\ \forall x\in\Omega_r,$$

and hence for every $K\in\mathbb{Z}$ and all $x\in\Omega_r\cap\widetilde{X}$,

$$\sum_{k=-\infty}^{K} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = \sum_{k=s_{rx}}^{K} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y), \tag{4.24}$$

i.e. the sum above is finite.

Observe that the fact that $\operatorname{supp}\psi_k(x,\cdot)\subset B(x,\tau 2^{-k})$ (see (4.4)) leads to the following conclusions:

(i) If $B(x,\tau 2^{-k})\subset E_{rk}$ for some $x\in E_{rk}\cap\widetilde{X}$, then

$$\begin{aligned}\int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f\,d\mu(y) &= \int_{B(x,\tau 2^{-k})} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y)\\ &= \int_X \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y).\end{aligned} \tag{4.25}$$

(ii) We have

$$\operatorname{supp}\Big(\int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big) \subset \{x : \operatorname{dist}(x,E_{rk})\le\tau 2^{-k}\}. \tag{4.26}$$

On the other hand, clearly $B(y,2\tau 2^{-k})\cap(\Omega_r\setminus\Omega_{r+1})\ne\emptyset$ for each $y\in E_{rk}$, and $\mathcal{N}_N(\tilde{\psi})\le c$. Hence (see (4.7)), $|\tilde{\psi}_k f(y)|\le c2^r$ for $y\in E_{rk}\cap\widetilde{X}$, and using (4.6) we get

$$\Big\|\int_E \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big\|_\infty \le c2^r, \qquad \forall E\subset E_{rk}. \tag{4.27}$$

Similarly,

$$\Big\|\int_E \varphi_k(\cdot,y)\varphi_k f(y)\,d\mu(y)\Big\|_\infty \le c2^r, \qquad \forall E\subset E_{rk}. \tag{4.28}$$

Two more estimates will be needed: for any measurable set $E \subset E_{rk}$,

$$\Big|\int_E \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big| \le cM_1(f\cdot \mathbb{1}_{\Omega_r})(x), \qquad \forall x \in \widetilde{X}, \tag{4.29}$$

$$\Big|\int_E \varphi_k(x,y)\tilde{\varphi}_k f(y)\,d\mu(y)\Big| \le cM_1(f\cdot \mathbb{1}_{\Omega_r})(x), \qquad \forall x \in \widetilde{X}, \tag{4.30}$$

where M_1 is the Hardy–Littlewood maximal operator (see (3.6)), and the constant c is independent of f, E, r, k, x. These two estimates follow readily by (4.4)–(4.6).

We record some of the main properties of F_r and F_{r,κ_0,κ_1} in the following

Lemma 4.2. (a) *The series in* (4.22) *and* (4.23) (*if* $\kappa_1 = \infty$) *converge in* L^2, *and hence the functions* F_r *and* F_{r,κ_0,κ_1} *are well defined.*

(b) *There exists a constant* $c > 0$ *such that for any* $r \in \mathbb{Z}$ *and integers* $-\infty \le \kappa_0 \le \kappa_1 \le \infty$,

$$\|F_r\|_\infty \le c2^r \quad \text{and} \quad \|F_{r,\kappa_0,\kappa_1}\|_\infty \le c2^r. \tag{4.31}$$

Moreover, there exists a set $Y_r \subset \widetilde{X}$ *such that* $\mu(X \setminus Y_r) = 0$ *and for every* $x \in Y_r$,

$$|F_r(x)| \le cM_1(f\cdot\mathbb{1}_{\Omega_r})(x) < \infty,\ |F_{r,\kappa_0,\kappa_1}(x)| \le cM_1(f\cdot\mathbb{1}_{\Omega_r})(x) < \infty. \tag{4.32}$$

Here M_1 *is the Hardy–Littlewood maximal operator, defined in* (3.6).

(c) *Also,*

$$F_r(x) = F_{r,\kappa_0,\kappa_1}(x) = 0, \qquad \forall x \in X \setminus \Omega_r,\ \forall r \in \mathbb{Z}. \tag{4.33}$$

(d) *We have*

$$E_{rk} \cap E_{r'k} = \emptyset \quad \text{if } r \ne r' \quad \text{and} \quad X = \bigcup_{r\in\mathbb{Z}} E_{rk}, \qquad \forall k \in \mathbb{Z}. \tag{4.34}$$

Proof. Identities (4.34) are obvious, and (4.33) follows readily from the definitions of F_r, F_{r,κ_0,κ_1} and (4.26).

Fix $r \in \mathbb{Z}$ and assume $\Omega_{r+1} \ne \emptyset$; the case when $\Omega_{r+1} = \emptyset$ is easier and will be omitted. We shall prove parts (a) and (b) of the lemma only for F_r; the proof for F_{r,κ_0,κ_1} is similar and will be omitted. We split the proof of (a)–(b) into two cases.

Case 1: *Estimates for* F_r *and* L^2-*convergence in* (4.22) *on* Ω_{r+1}. We next show that $F_r(x)$ is well defined pointwise by (4.22) on $\Omega_{r+1} \cap \widetilde{X}$, and the left estimates in (4.31)–(4.32) hold on $\Omega_{r+1} \cap \widetilde{X}$. In fact, it will be shown that the sum in (4.22) is finite for every $x \in \Omega_{r+1} \cap \widetilde{X}$. Set

$$\begin{aligned} U_k &:= \{x \in \Omega_r : \operatorname{dist}(x, \Omega_r^c) > 2\tau 2^{-k}\},\\ V_k &:= \{x \in \Omega_{r+1} : \operatorname{dist}(x, \Omega_{r+1}^c) > 2\tau 2^{-k}\}. \end{aligned} \tag{4.35}$$

Then $E_{rk} = U_k \setminus V_k$ (see (4.21)).

From (4.26) it follows that

$$F_r(x) = 0 \quad \text{for } x \in X \setminus \bigcup_{k\in\mathbb{Z}} \{y : \mathrm{dist}(y, E_{rk}) < \tau 2^{-k}\}.$$

We next estimate $|F_r(x)|$ for

$$x \in \Big[\bigcup_{k\in\mathbb{Z}} \{y : \mathrm{dist}(y, E_{rk}) < \tau 2^{-k}\}\Big] \cap \Omega_{r+1} \cap \widetilde{X}.$$

For any such x there exist $\nu, \ell \in \mathbb{Z}$ such that

$$x \in (U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu). \tag{4.36}$$

As $\Omega_{r+1} \subset \Omega_r$ we have $V_k \subset U_k$, implying $(U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu) = \emptyset$ if $\nu < \ell$. We consider two subcases depending on whether $\nu \ge \ell + 3$ or $\ell \le \nu \le \ell + 2$.

(a) Let $\nu \ge \ell + 3$. We claim that (4.21) and (4.36) yield

$$B(x, \tau 2^{-k}) \cap E_{rk} = \emptyset \quad \text{for } k \ge \nu + 2 \text{ or } k \le \ell - 1. \tag{4.37}$$

Indeed, if $k \ge \nu + 2$, then $E_{rk} \subset \Omega_r \setminus V_{\nu+2}$, which implies (4.37), while if $k \le \ell - 1$, then $E_{rk} \subset U_{\ell-1}$, again implying (4.37).

We also claim that

$$B(x, \tau 2^{-k}) \subset E_{rk} \quad \text{for } \ell + 2 \le k \le \nu - 1. \tag{4.38}$$

Indeed, if $\ell + 2 \le k \le \nu - 1$, then

$$(U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu) \subset (U_{k-1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_{k+1}) \subset U_{k-1} \setminus V_{k+1},$$

which implies (4.38).

From (4.25)–(4.26) and (4.37)–(4.38) it follows that

$$\begin{aligned} F_r(x) &= \sum_{k=\ell}^{\nu+1} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = \sum_{k=\ell}^{\ell+1} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) \\ &\quad + \sum_{k=\ell+2}^{\nu-2} \int_X \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) + \sum_{k=\nu-1}^{\nu+1} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y). \end{aligned}$$

However, using the notation from (4.8),

$$\sum_{k=\ell+2}^{\nu-2} \int_X \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = \sum_{k=\ell+2}^{\nu-2} \psi_k\tilde{\psi}_k f(x) = Q_{\ell+2,\nu-2} f(x).$$

Since $\mathrm{dist}(x, \Omega_r \setminus \Omega_{r+1}) \le 2\tau 2^{-(\nu-2)}$ and $B(x, 2\tau 2^{-(\ell+2)}) \subset \Omega_r$ it follows by Lemma 4.1 that $|Q_{\ell+2,\nu-2} f(x)| \le c2^r$ and $|Q_{\ell+2,\nu-2} f(x)| \le cM_1(f \cdot \mathbb{1}_{\Omega_r})(x)$. We use the above, (4.27)–(4.28), and (4.29)–(4.30) to obtain $|F_r(x)| \le c2^r$ and $|F_r(x)| \le cM_1(f \cdot \mathbb{1}_{\Omega_r})(x)$.

(b) Let $\ell \le \nu \le \ell+2$. We have

$$F_r(x) = \sum_{k=\ell}^{\nu+1} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = \sum_{k=\ell}^{\ell+3} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y).$$

We use (4.27) and (4.29) to estimate each of these four integrals and obtain again $|F_r(x)| \le c2^r$ and $|F_r(x)| \le cM_1(f\cdot \mathbb{1}_{\Omega_r})(x)$.

The sum in (4.22) is finite for every $x \in \Omega_{r+1} \cap \widetilde{X}$ and using the fact that $\|F_r\|_{L^\infty(\Omega_{r+1})} \le c2^r$ and $\mu(\Omega_{r+1}) < \infty$ (see (4.20)) it follows that the series in (4.22) converges in $L^2(\Omega_{r+1})$.

Case 2: *Estimates for F_r and L^2-convergence in* (4.22) *on $\Omega_r \setminus \Omega_{r+1}$.* The following two estimates will play an important rôle: for any $K \in \mathbb{Z}$,

$$\Big\| \sum_{k=-\infty}^{K} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big\|_{L^\infty(\Omega_r\setminus\Omega_{r+1})} \le c2^r \tag{4.39}$$

and

$$\Big| \sum_{k=-\infty}^{K} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big| \le cM_1(f\cdot\mathbb{1}_{\Omega_r})(x),\ \forall x \in (\Omega_r\setminus\Omega_{r+1})\cap\widetilde{X}, \tag{4.40}$$

where $c > 0$ is a constant independent of r and K. Write

$$S_K(x) := \sum_{k=-\infty}^{K} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = \sum_{k=s_{rx}}^{K} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y).$$

Clearly,

$$\Omega_r\setminus\Omega_{r+1} = \bigcup_{\ell\in\mathbb{Z}} (U_{\ell+1}\setminus U_\ell)\cap(\Omega_r\setminus\Omega_{r+1}). \tag{4.41}$$

Let $x \in (\Omega_r\setminus\Omega_{r+1})\cap\widetilde{X}$. Then $x \in (U_{\ell+1}\setminus U_\ell)\cap(\Omega_r\setminus\Omega_{r+1})$ for some $\ell\in\mathbb{Z}$. Just as in the proof of (4.37) we have $B(x,\tau 2^{-k})\cap E_{rk} = \emptyset$ for $k \le \ell-1$, and as in the proof of (4.38) we have

$$(U_{\ell+1}\setminus U_\ell)\cap(\Omega_r\setminus\Omega_{r+1}) \subset U_{k-1}\setminus V_{k+1} \quad \text{for } k \ge \ell+2,$$

which implies $B(x,\tau 2^{-k}) \subset E_{rk}$ for $k \ge \ell+2$.

Assume that $K > \ell+1$. We use the above and (4.25)–(4.26) to obtain

$$\begin{aligned} S_K(x) &= \sum_{k=\ell}^{K} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) \\ &= \sum_{k=\ell}^{\ell+1} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) + \sum_{k=\ell+2}^{K} \int_{X} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y). \end{aligned}$$

For the last sum we have

$$\sum_{k=\ell+2}^{K} \int_X \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = \sum_{k=\ell+2}^{K} \psi_k\tilde{\psi}_k f(x) = Q_{\ell+2,K} f(x).$$

The representation of $S_K(x)$ for $K \le \ell+1$ is similar. Now, Lemma 4.1, (4.27)–(4.28), and (4.29)–(4.30) yield (4.39) and (4.40).

We next prove the convergence of the series in the definition of F_r in (4.22) in $L^2(W_\ell)$, where

$$W_\ell := (U_{\ell+1} \setminus U_\ell) \cap (\Omega_r \setminus \Omega_{r+1}), \quad \ell \in \mathbb{Z}. \tag{4.42}$$

From the above we know that for all $k \ge \ell+2$ and $x \in W_\ell \cap \widetilde{X}$ we have

$$\int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = \int_X \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = \psi_k\tilde{\psi}_k f(x).$$

Therefore, for every $K \ge \ell+2$,

$$\begin{aligned}\Big\|\sum_{k=K}^{\infty} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big\|_{L^2(W_\ell)} &= \Big\|\sum_{k=K}^{\infty} \psi_k\tilde{\psi}_k f\Big\|_{L^2(W_\ell)} \\ &\le \Big\|\sum_{k=K}^{\infty} \psi_k\tilde{\psi}_k f\Big\|_{L^2(X)},\end{aligned}$$

and hence from the convergence in (4.16) or (4.17) it follows that

$$\lim_{K\to\infty}\Big\|\sum_{k=K}^{\infty} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big\|_{L^2(W_\ell)} = 0, \quad \forall \ell \in \mathbb{Z}. \tag{4.43}$$

We are now prepared to prove the convergence in $L^2(\Omega_r \setminus \Omega_{r+1})$ of the series in the definition of F_r in (4.22), which is the same as the convergence of $\{S_K\}$ in $L^2(\Omega_r\setminus\Omega_{r+1})$. From (4.41) and (4.42) it follows that $\Omega_r\setminus\Omega_{r+1} = \bigcup_{\ell\in\mathbb{Z}} W_\ell$ and observe that the sets $\{W_\ell\}$ are disjoint. Hence

$$|\Omega_r \setminus \Omega_{r+1}| = \sum_{\ell\in\mathbb{Z}} |W_\ell|.$$

Fix $\varepsilon > 0$. From (4.20) we know that $|\Omega_r| < \infty$, and hence there exists $M \in \mathbb{Z}$ such that

$$|(\Omega_r \setminus \Omega_{r+1}) \setminus U_M| + |(\Omega_r \setminus \Omega_{r+1}) \cap U_{-M}| = \sum_{|\ell|\ge M} |W_\ell| < \varepsilon.$$

From this and (4.39) it follows that

$$\|S_K\|_{L^2[\Omega_r\setminus\Omega_{r+1})\setminus U_M]} + \|S_K\|_{L^2[\Omega_r\setminus\Omega_{r+1})\cap U_{-M}]} \le c2^r\varepsilon^{1/2}, \quad \forall K \in \mathbb{Z}. \tag{4.44}$$

Clearly,

$$\Omega_r \setminus \Omega_{r+1} = \bigcup_{\ell=-M}^{M} W_\ell \cup [(\Omega_r \setminus \Omega_{r+1}) \setminus U_M] \cup [(\Omega_r \setminus \Omega_{r+1}) \cap U_{-M}].$$

This, (4.43), and (4.44) imply the convergence of the series in (4.22) in $L^2(\Omega_r \setminus \Omega_{r+1})$. To see this, one simply shows that $\|S_{K_1} - S_{K_2}\|_{L^2(\Omega_r\setminus\Omega_{r+1})} \to 0$ as $K_1, K_2 \to \infty$.

Since the series in (4.22) converges in $L^2(\Omega_r \setminus \Omega_{r+1})$, it follows that (see e.g. [9, Theorem 3.12]) there exists a sequence $K_1 < K_2 < \cdots$ such that

$$F_r(x) = \lim_{j\to\infty} \sum_{k=-\infty}^{K_j} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\, d\mu(y) \quad \text{for a.a. } x \in \Omega_r \setminus \Omega_{r+1}.$$

This coupled with (4.39) and (4.40) yields

$$\|F_r\|_{L^\infty(\Omega_r\setminus\Omega_{r+1})} \le c2^r \quad \text{and} \quad |F_r(x)| \le cM_1(f\cdot \mathbb{1}_{\Omega_r})(x) \text{ a.e. on } \Omega_r \setminus \Omega_{r+1}.$$

The proof of Lemma 4.2 is complete. ∎

For convenience, we define $F_r := 0$ whenever $\Omega_r = \emptyset$, $r \in \mathbb{Z}$.

Observe that from (4.34) it follows that, for $x \in \widetilde{X}$,

$$\psi_k\tilde{\psi}_k f(x) = \int_X \psi_k(x,y)\tilde{\psi}_k f(y)\, d\mu(y) = \sum_{r\in\mathbb{Z}} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\, d\mu(y),$$

and using (4.17) and the definition of F_r in (4.22) we obtain $f = \sum_{r\in\mathbb{Z}} F_r$. We next present the needed justification for this identity.

Lemma 4.3. *We have*

$$f = \sum_{r\in\mathbb{Z}} F_r \quad (\textit{convergence in } L^2). \tag{4.45}$$

Proof. We first show that

$$\Big\| f - \sum_{r=-R}^{R} \sum_{k=-K}^{K} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\, d\mu(y) \Big\|_{L^2} \to 0 \quad \text{as } R, K \to \infty. \tag{4.46}$$

Note that (4.17) implies

$$\Big\| f - \sum_{k=-K}^{K} \int_X \psi_k(\cdot,y)\tilde{\psi}_k f(y)\, d\mu(y) \Big\|_{L^2} \to 0 \quad \text{as } K \to \infty. \tag{4.47}$$

From (4.34) it follows that

$$\int_X \psi_k(x,y)\tilde{\psi}_k f(y)\, d\mu(y) = \sum_{r\in\mathbb{Z}} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\, d\mu(y), \quad x \in \widetilde{X}. \tag{4.48}$$

Further, using (4.20) and (4.27) we get

$$\Big\| \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\, d\mu(y) \Big\|_{L^2(X)} \le c2^r|\Omega_r|^{1/2} \le c\|f\|_{H^p}^p 2^{r(1-p/2)},$$

and hence

$$\Big\| \sum_{r<-R} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big\|_{L^2(X)} \le c\|f\|_{H^p}^p \sum_{r<-R} 2^{r(1-p/2)} \le c\|f\|_{H^p}^p 2^{-R(1-p/2)} \to 0 \quad \text{as } R\to\infty. \tag{4.49}$$

On the other hand, denoting

$$U_{mk} := \{x\in \Omega_m : \operatorname{dist}(x,\Omega_m^c) > 2\tau 2^{-k}\} = \bigcup_{r\ge m} E_{rk} \tag{4.50}$$

and using (4.4)–(4.6) we obtain, for $x\in \widetilde{X}$,

$$\Big|\sum_{r\ge m} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big| = \Big|\int_{U_{mk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big| \le c|B(x,2^{-k})|^{-1} \int_{U_{mk}\cap B(x,\tau 2^{-k})} |f(y)|\,d\mu(y) \le cM_1(f\cdot \mathbb{1}_{U_{mk}})(x).$$

Here M_1 is the Hardy–Littlewood maximal operator (see (3.6)), and we have used the fact that $|B(x,\tau 2^{-k})| \le c_0\tau^d |B(x,2^{-k})|$, applying (1.2). Therefore,

$$\Big\|\sum_{r\ge m} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big\|_{L^2} \le c\|M_1(f\cdot \mathbb{1}_{U_{mk}})\|_{L^2} \le c\|f\|_{L^2(U_{mk})} \le c\|f\|_{L^2(\Omega_m)} \to 0 \quad \text{as } m\to\infty, \tag{4.51}$$

where we have used the maximal inequality (3.7) and the fact that $|\Omega_m|\to 0$ as $m\to\infty$. Clearly, (4.48), (4.49), and (4.51) yield

$$\Big\|\int_X \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y) - \sum_{r=-R}^{R} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big\|_{L^2} \to 0 \quad \text{as } R\to\infty.$$

This coupled with (4.47) implies (4.46).

Our second step is to show that

$$\Big\|\sum_{r\in\mathbb{Z}} F_r - \sum_{r=-R}^{R}\sum_{k=-K}^{K} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y)\Big\|_{L^2} \to 0 \quad \text{as } R,K\to\infty. \tag{4.52}$$

We first use (4.20), (4.31), and (4.33) to obtain

$$\Big\|\sum_{r<-R} F_r\Big\|_{L^2} \le \sum_{r<-R} \|F_r\|_{L^2} \le c\sum_{r<-R} 2^r|\Omega_r|^{1/2} \le c\|f\|_{H^p}^{p/2} \sum_{r<-R} 2^{r(1-p/2)} \le c\|f\|_{H^p}^{p/2} 2^{-R(1-p/2)} \to 0 \quad \text{as } R\to\infty. \tag{4.53}$$

We next estimate $\| \sum_{r\ge m} F_r\|_{L^2}$. Observe that by (4.33) it follows that for any $m \in \mathbb{Z}$ we have $\sum_{r\ge m} F_r(x) = 0$ for $x \in X \setminus \Omega_m$. We claim that for any $m \in \mathbb{Z}$,

$$\Big|\sum_{r\ge m} F_r(x)\Big| \le cM_1(f \cdot \mathbb{1}_{\Omega_m})(x) \quad \text{for a.a. } x \in \Omega_m. \tag{4.54}$$

To prove this, just as in (4.50) we let

$$U_{rk} := \{x \in \Omega_r : \operatorname{dist}(x, \Omega_r^c) > 2\tau 2^{-k}\}.$$

Set $Y := \bigcap_{r\in\mathbb{Z}} Y_r$, where the sets Y_r are from Lemma 4.2. Clearly $Y \subset \widetilde{X}$ and $\mu(X \setminus Y) = 0$.

Let $x \in \Omega_m \cap Y$. Then $x \in (\Omega_r \setminus \Omega_{r+1}) \cap Y$ for some $r \ge m$. By (4.33) it follows that

$$\sum_{r'\ge m} F_{r'}(x) = \sum_{r'=m}^{r} F_{r'}(x).$$

Clearly, $x \in (U_{r,\ell+1} \setminus U_{r\ell}) \cap (\Omega_r \setminus \Omega_{r+1})$ for some $\ell \in \mathbb{Z}$. Also, from the definition of E_{rk} in (4.21) we have $U_{mk} = \bigcup_{r\ge m} E_{rk}$. In light of (4.22) this implies

$$\sum_{r'=m}^{r} F_{r'}(x) = F_r(x) + \sum_{k=-\infty}^{\ell} \int_{U_{mk}} \psi_k(x,y)\tilde{\psi}_k f(y)\, d\mu(y).$$

From (4.32) we have $|F_r(x)| \le cM_1(f \cdot \mathbb{1}_{\Omega_m})(x)$, and just as in the proof of Lemma 4.2 one shows that

$$\Big|\sum_{k=-\infty}^{\ell} \int_{U_{mk}} \psi_k(x,y)\tilde{\psi}_k f(y)\, d\mu(y)\Big| \le cM_1(f \cdot \mathbb{1}_{\Omega_m})(x).$$

Putting the above together we obtain (4.54). In turn, (4.54) and the maximal inequality (3.7) imply

$$\Big\|\sum_{r\ge m} F_r\Big\|_{L^2} \le c\|M_1(f \cdot \mathbb{1}_{\Omega_m})\|_{L^2} \le c\|f\|_{L^2(\Omega_m)} \to 0 \quad \text{as } m \to \infty, \tag{4.55}$$

where we have used the fact that $|\Omega_m| \to 0$ as $m \to \infty$.

As was shown in Lemma 4.2, the series in the definition of F_r in (4.22) converges in L^2, and hence for any $r \in \mathbb{Z}$,

$$\Big\|F_r - \sum_{k=-\infty}^{K} \int_{E_{rk}} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\, d\mu(y)\Big\|_{L^2} \to 0 \quad \text{as } K \to \infty.$$

This, (4.53), and (4.55) imply (4.52). In turn, (4.46) and (4.52) yield (4.45). ∎

We next break each function F_r into atoms. To this end we need a Whitney type cover for Ω_r.

LEMMA 4.4. *Assume that Ω is an open subset of X, $\Omega \neq X$, and denote $\rho(x) := \operatorname{dist}(x, \Omega^c)$. Then there exist a constant $K > 0$ ($K = 70^d c_0^2$ will do) and a sequence $\{\xi_j\}_{j\in\mathbb{N}}$ of points in Ω with the following properties, where $\rho_j := \operatorname{dist}(\xi_j, \Omega^c)$:*

(a) $\Omega = \bigcup_{j\in\mathbb{N}} B(\xi_j, \rho_j/2)$.
(b) $\{B(\xi_j, \rho_j/5)\}$ *are disjoint.*
(c) *If* $B(\xi_j, 3\rho_j/4) \cap B(\xi_\nu, 3\rho_\nu/4) \neq \emptyset$, *then* $7^{-1}\rho_\nu \leq \rho_j \leq 7\rho_\nu$.
(d) *For every* $j \in \mathbb{N}$ *there are at most* K *balls* $B(\xi_\nu, 3\rho_\nu/4)$ *intersecting* $B(\xi_j, 3\rho_j/4)$.

Variants of this lemma are well known and frequently used. To prove it one selects $\{B(\xi_j, \rho(\xi_j)/5)\}_{j\in\mathbb{N}}$ to be a maximal disjoint subcollection of $\{B(x, \rho(x)/5)\}_{x\in\Omega}$, and then properties (a)–(d) follow readily (see [12, pp. 15–16]. For completeness we give the proof in the appendix.

We apply Lemma 4.4 to each set $\Omega_r \neq \emptyset$. Fix $r \in \mathbb{Z}$ and assume $\Omega_r \neq \emptyset$. Denote by $B_j := B(\xi_j, \rho_j/2)$, $j = 1, 2, \ldots$, the balls given by Lemma 4.4 applied to Ω_r, with the additional assumption that these balls are ordered so that $\rho_1 \geq \rho_2 \geq \cdots$. We shall adhere to the notation of Lemma 4.4. We shall also use the more compact notation $\mathcal{B}_r := \{B_j\}_{j\in\mathbb{N}}$ for the set of balls covering Ω_r.

For each ball $B \in \mathcal{B}_r$ and $k \in \mathbb{Z}$ we define

$$E_{rk}^B := E_{rk} \cap \{x \in X : \operatorname{dist}(x, B) < 2\tau 2^{-k}\} \quad \text{if } B \cap E_{rk} \neq \emptyset \tag{4.56}$$

and set $E_{rk}^B := \emptyset$ if $B \cap E_{rk} = \emptyset$.

We also define, for $\ell = 1, 2, \ldots,$

$$R_{rk}^{B_\ell} := E_{rk}^{B_\ell} \setminus \bigcup_{\nu>\ell} E_{rk}^{B_\nu}, \tag{4.57}$$

$$F_{B_\ell} := \sum_{k\in\mathbb{Z}} \int_{R_{rk}^{B_\ell}} \psi_k(\cdot, y)\tilde{\psi}_k f(y)\, d\mu(y), \qquad G_{B_\ell} := L^{-n} F_{B_\ell}, \tag{4.58}$$

where the convergence of the series is in $L^2(X)$.

LEMMA 4.5. (a) *For any $\ell \geq 1$ the series in* (4.58) *converges in L^2 and hence $F_{B_\ell} \in L^2$ is well defined. Furthermore, G_{B_ℓ} is well defined.*

(b) *There exists a constant $c_\sharp > 0$ such that for every $\ell \geq 1$,*

$$\|F_{B_\ell}\|_\infty \leq c_\sharp 2^r, \qquad \|L^m G_{B_\ell}\|_\infty \leq c_\sharp 2^r \rho_\ell^{2(n-m)} \quad \text{for } m = 0, 1, \ldots, n, \tag{4.59}$$

$$\operatorname{supp} F_{B_\ell} \subset 7B_\ell, \qquad \operatorname{supp} L^m G_{B_\ell} \subset 7B_\ell \quad \text{for } m = 0, \ldots, n. \tag{4.60}$$

(c) *For any $k \in \mathbb{Z}$*

$$E_{rk} = \bigcup_{\ell\geq 1} R_{rk}^{B_\ell} \quad \text{and} \quad R_{rk}^{B_\ell} \cap R_{rk}^{B_m} = \emptyset \quad \text{if } \ell \neq m. \tag{4.61}$$

(d) *We have*

$$F_r = \sum_{B\in\mathcal{B}_r} F_B \quad (\textit{convergence in } L^2). \tag{4.62}$$

To prove this lemma we need some preparation.

LEMMA 4.6. *For an arbitrary measurable set $S \subset X$ let*

$$S_k := \{x \in X : \operatorname{dist}(x,S) < 2\tau 2^{-k}\}$$

and

$$F_S := \sum_{k\ge\kappa_0} \int_{E_{rk}\cap S_k} \psi_k(\cdot,y)\tilde{\psi}_k f(y)\,d\mu(y) \tag{4.63}$$

for some $\kappa_0 \ge -\infty$, where the convergence is in $L^2(X)$. Then F_S is well defined, $F_S \in L^2(X)\cap L^\infty(X)$, and $\|F_S\|_\infty \le c2^r$, where $c > 0$ is a constant independent of S and κ_0.

Proof. From (4.26) it follows that $F_S(x) = 0$ if $\operatorname{dist}(x,S) \ge 3\tau 2^{-\kappa_0}$ or $x \in X \setminus \Omega_r$.

Let $x \in S \cap \widetilde{X}$. Clearly, $B(x,\tau 2^{-k}) \subset S_k$ for every k, and hence

$$\int_{E_{rk}\cap B(x,\tau 2^{-k})} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) = \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y).$$

Therefore, $F_S = F_{r,\kappa_0,\infty}$ on S (see (4.23)). On account of Lemma 4.2 the series in (4.63) converges in $L^2(S)$, $F_S \in L^2(S)\cap L^\infty(S)$, and $\|F_S\|_{L^\infty(S)} = \|F_{r,\kappa_0,\infty}\|_\infty \le c2^r$.

We now consider F_S on $X\setminus S$. Let $x \in (S_\ell \setminus S_{\ell+1})\cap Y_r$ for some $\ell \ge \kappa_0$, where the set $Y_r \subset \widetilde{X}$ is from Lemma 4.2. Then $B(x,\tau 2^{-k}) \subset S_k$ whenever $\kappa_0 \le k \le \ell-1$, and $B(x,\tau 2^{-k})\cap S_k = \emptyset$ if $k \ge \ell+2$. Therefore,

$$\begin{aligned} F_S(x) &= \sum_{k=\kappa_0}^{\ell-1} \int_{E_{rk}} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) + \sum_{k=\ell}^{\ell+1} \int_{E_{rk}\cap S_k} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y) \\ &= F_{r,\kappa_0,\ell-1}(x) + \sum_{k=\ell}^{\ell+1} \int_{E_{rk}\cap S_k} \psi_k(x,y)\tilde{\psi}_k f(y)\,d\mu(y), \end{aligned}$$

where we have used the notation from (4.23). By Lemma 4.2 and (4.27) it follows that $|F_S(x)| \le c2^r$.

We finally consider the case when $2\tau 2^{-\kappa_0} \le \operatorname{dist}(x,S) < 3\tau 2^{-\kappa_0}$ and $x \in \widetilde{X}$. Then we have $F_S(x) = \int_{E_{r\kappa_0}\cap S_{\kappa_0}} \psi_{\kappa_0}(x,y)\tilde{\psi}_{\kappa_0} f(y)\,d\mu(y)$, and the estimate $|F_S(x)| \le c2^r$ is immediate from (4.27).

Hence, $F_S(x)$ is well defined for $x \in (X\setminus S)\cap Y_r$ and $\|F_S\|_{L^\infty(X\setminus S)} \le c2^r$. Furthermore, since $F_S(x) = 0$ for $x \in X\setminus\Omega_r$ and $|\Omega_r| < \infty$, it follows that the series in (4.63) converges in $L^2(X\setminus S)$. ∎

Proof of Lemma 4.5. By Lemma 4.4 we have $\Omega_r = \bigcup_{\ell\in\mathbb{N}} B_\ell$, and then (4.61) is immediate from (4.56) and (4.57).

Fix $\ell \ge 1$. Observe that by Lemma 4.4, $B_\ell \subset \{x : \operatorname{dist}(x, \Omega_r^c) < 2\rho_\ell\}$, and hence $E_{rk}^{B_\ell} := \emptyset$ if $2\tau 2^{-k} \ge 2\rho_\ell$. Define $k_0 := \min\{k : \tau 2^{-k} < \rho_\ell\}$. Hence $\rho_\ell/2 \le \tau 2^{-k_0} < \rho_\ell$. Consequently,

$$F_{B_\ell} = \sum_{k\ge k_0} \int_{R_{rk}^{B_\ell}} \psi_k(\cdot, y)\tilde{\psi}_k f(y)\, d\mu(y), \tag{4.64}$$

where the convergence is in L^2 and will be validated later on.

Using (4.4) we get $\operatorname{supp} F_{B_\ell} \subset B(\xi_\ell, (7/2)\rho_\ell) = 7B_\ell$, which confirms the left-hand inclusion in (4.60).

With $\ell \ge 1$ being fixed, we let $\{B_j : j \in \mathcal{J}\}$ denote the set of all balls $B_j = B(\xi_j, \rho_j/2)$ such that $j > \ell$ and

$$B(\xi_j, 3\rho_j/4) \cap B(\xi_\ell, 3\rho_\ell/4) \ne \emptyset.$$

By Lemma 4.4 it follows that $\#\mathcal{J} \le K$ and $7^{-1}\rho_\ell \le \rho_j \le \rho_\ell$ for $j \in \mathcal{J}$. Define

$$k_1 := \min\bigl\{k : 2\tau 2^{-k} < 4^{-1} \min\{\rho_j : j \in \mathcal{J} \cup \{\ell\}\}\bigr\}. \tag{4.65}$$

From this definition and $\tau 2^{-k_0} < \rho_\ell$ we obtain

$$2\tau 2^{-k_1} \ge 8^{-1} \min\bigl\{\rho_j : j \in \mathcal{J} \cup \{\ell\}\bigr\} > 8^{-2}\rho_\ell > 8^{-2}\tau 2^{-k_0} \;\Rightarrow\; k_1 \le k_0 + 7. \tag{4.66}$$

Clearly, from (4.65),

$$B(\xi_j, \rho_j/2 + 2\tau 2^{-k}) \subset B(\xi_j, 3\rho_j/4), \quad \forall k \ge k_1,\ \forall j \in \mathcal{J} \cup \{\ell\}. \tag{4.67}$$

Denote $S := \bigcup_{j\in\mathcal{J}} B_j$ and $\tilde{S} := \bigcup_{j\in\mathcal{J}} B_j \cup B_\ell = S \cup B_\ell$. As in Lemma 4.6 we set

$$S_k := \{x \in X : \operatorname{dist}(x, S) < 2\tau 2^{-k}\}, \quad \tilde{S}_k := \{x \in X : \operatorname{dist}(x, \tilde{S}) < 2\tau 2^{-k}\}.$$

It readily follows from the definition of k_1 in (4.65) and (4.57) that

$$R_{rk}^{B_\ell} := E_{rk}^{B_\ell} \setminus \bigcup_{\nu>\ell} E_{rk}^{B_\nu} = (E_{rk} \cap \tilde{S}_k) \setminus (E_{rk} \cap S_k) \quad \text{for } k \ge k_1. \tag{4.68}$$

Set

$$F_S := \sum_{k\ge k_1} \int_{E_{rk}\cap S_k} \psi_k(\cdot, y)\tilde{\psi}_k f(y)\, d\mu(y), \tag{4.69}$$

$$F_{\tilde{S}} := \sum_{k\ge k_1} \int_{E_{rk}\cap \tilde{S}_k} \psi_k(\cdot, y)\tilde{\psi}_k f(y)\, d\mu(y). \tag{4.70}$$

By Lemma 4.6 it follows that the series in (4.69)–(4.70) converge in L^2, and hence F_S and $F_{\tilde{S}}$ are well defined. Also, just as in the proof of Lemma 4.2, using (4.68) one shows that the series in (4.64) converges in L^2, and hence F_{B_ℓ} is well defined as well. Now, from (4.68) and the fact that $S \subset \tilde{S}$ we get

$$F_{B_\ell} = F_{\tilde{S}} - F_S + \sum_{k_0 \le k < k_1} \int_{R_{rk}^{B_\ell}} \psi_k(\cdot, y) \tilde{\psi}_k f(y)\, d\mu(y).$$

Applying Lemma 4.6 to the functions F_S and $F_{\tilde{S}}$ from (4.69)–(4.70) we deduce $\|F_S\|_\infty \le c2^r$ and $\|F_{\tilde{S}}\|_\infty \le c2^r$. On the other hand, from (4.66) we have $k_1 - k_0 \le 7$. We use the above and also estimate each of the (at most 7) integrals above using (4.27) to conclude that $\|F_{B_\ell}\|_\infty \le c2^r$ as claimed.

By (4.58) we have $G_{B_\ell} := L^{-n} F_{B_\ell}$. We next show that for any $0 \le m < n$ the function $L^m G_{B_\ell}$ is well defined and $\|L^m G_{B_\ell}\|_\infty \le c2^r \rho_\ell^{2(n-m)}$, which is the right-hand estimate in (4.59). By the definition

$$\begin{aligned} L^m G_{B_\ell} &= L^{-(n-m)} F_{B_\ell} \\ &= \sum_{k \ge k_0} \int_{R_{rk}^{B_\ell}} [L^{-(n-m)} \psi(2^{-k}\sqrt{L})](\cdot, y) \tilde{\psi}(2^{-k}\sqrt{L}) f(y)\, d\mu(y), \end{aligned} \tag{4.71}$$

where we have used the equality

$$L^{-\nu}[\psi(2^{-k}\sqrt{L})(x,y)] = [L^{-\nu}\psi(2^{-k}\sqrt{L})](x,y)$$

for a.a. $x, y \in X$ and $\nu = 1, \dots, n$, which is a consequence of Lemma 2.4.

To justify the convergence in (4.71) we let $g(\lambda) := \lambda^{-2(n-m)}\psi(\lambda)$. Then

$$L^{-(n-m)}\psi(2^{-k}\sqrt{L}) = 2^{-2k(n-m)} g(2^{-k}\sqrt{L}).$$

From (4.5) we have $\operatorname{supp}[L^{-(n-m)}\psi(2^{-k}\sqrt{L})](x,\cdot) \subset B(x, \tau 2^{-k})$, and by Theorem 2.3 we get

$$|[L^{-(n-m)}\psi(2^{-k}\sqrt{L})](x,y)| \le c2^{-2k(n-m)} |B(x, 2^{-k})|^{-1}, \quad \forall x, y \in \widetilde{X}.$$

On the other hand, by (4.27),

$$|\tilde{\psi}(2^{-k}\sqrt{L}) f(y)| \le c2^r \quad \text{for } y \in R_{rk}^{B_\ell} \cap \widetilde{X} \subset E_{rk} \cap \widetilde{X}.$$

Putting the above together we deduce that for almost all $x \in X$,

$$\begin{aligned} &\Big| \int_{R_{rk}^{B_\ell}} [L^{-(n-m)}\psi(2^{-k}\sqrt{L})](x,y) \tilde{\psi}(2^{-k}\sqrt{L}) f(y)\, d\mu(y) \Big| \\ &\qquad \le c2^r 2^{-2k(n-m)} \int_{B(x,\tau 2^{-k})} |B(x, 2^{-k})|^{-1}\, d\mu(y) \le c2^r 2^{-2k(n-m)}, \end{aligned}$$

where we have used the fact that $B|(x, \tau 2^{-k})| \le c_0 \tau^2 |B(x, 2^{-k})|$ by (1.2). Hence,

$$\|L^m G_{B_\ell}\|_\infty \le c2^r \sum_{k \ge k_0} 2^{-2k(n-m)} \le c2^r 2^{-2k_0(n-m)} \le c2^r \rho_\ell^{2(n-m)}$$

as claimed (see the right-hand inequality of (4.59)).

Just as for F_{B_ℓ}, from (4.5) and (4.71) it follows that $\operatorname{supp} G_{B_\ell} \subset 7B_\ell$. Furthermore, from the above it also follows that the series in (4.71) converges in $L^\infty(7B_\ell)$, and hence in L^2. Therefore, G_{B_ℓ} is well defined. This completes the proof of Lemma 4.5. ∎

We are now in a position to complete the proof of Theorem 1.4. For every ball $B \in \mathcal{B}_r$, $r \in \mathbb{Z}$, provided $\Omega_r \neq \emptyset$, we define $B^\star := 7B$,

$$a_B := c_\sharp^{-1}|B^\star|^{-1/p}2^{-r}F_B, \qquad b_B := c_\sharp^{-1}|B^\star|^{-1/p}2^{-r}G_B,$$

and $\lambda_B := c_\sharp |B^\star|^{1/p}2^r$, where $c_\sharp > 0$ is the constant from (4.59). By (4.60) we have $\operatorname{supp} a_B \subset B^\star$ and $\operatorname{supp} L^m b_B \subset B^\star$, $m = 0, \dots, n$, and by (4.59),

$$\|a_B\|_\infty \le c_\sharp^{-1}|B^\star|^{-1/p}2^{-r}\|F_B\|_\infty \le |B^\star|^{-1/p}.$$

From (4.58) it follows that $L^n b_B = a_B$, and assuming that $B = B(\xi_\ell, \rho_\ell/2)$ we obtain, using (4.59),

$$\begin{aligned}\|L^m b_B\|_\infty &\le c_\sharp^{-1}|B^\star|^{-1/p}2^{-r}\|L^m G_B\|_\infty \\ &\le \rho_\ell^{2(n-m)}|B^\star|^{-1/p} \le r_{B^\star}^{2(n-m)}|B^\star|^{-1/p}.\end{aligned}$$

Therefore, each a_B is an atom for H^p.

We set $\mathcal{B}_r := \emptyset$ if $\Omega_r = \emptyset$. Now, from the above, (4.45), and Lemma 4.5 we infer that

$$f = \sum_{r\in\mathbb{Z}} F_r = \sum_{r\in\mathbb{Z}}\sum_{B\in\mathcal{B}_r} F_B = \sum_{r\in\mathbb{Z}}\sum_{B\in\mathcal{B}_r} \lambda_B a_B,$$

where the convergence is in L^2, and

$$\sum_{r\in\mathbb{Z}}\sum_{B\in\mathcal{B}_r} |\lambda_B|^p \le c\sum_{r\in\mathbb{Z}} 2^{pr}\sum_{B\in\mathcal{B}_r}|B| = c\sum_{r\in\mathbb{Z}} 2^{pr}|\Omega_r| \le c\|f\|_{H^p}^p,$$

which is the claimed atomic decomposition of $f \in H^p$. Above we have used (4.19) and the fact that $|B^\star| = |7B| \le c_0 7^d|B|$. ∎

4.2. Proof of the embedding $H_A^p \subset H^p$ in the noncompact case. We next show that if $f \in H_A^p$, then $f \in H^p$ and $\|f\|_{H^p} \le c\|f\|_{H_A^p}$. To this end we need the following

LEMMA 4.7. *For any atom a and $0 < p \le 1$, we have*

$$\|a\|_{H^p} \le c < \infty. \tag{4.72}$$

Proof. Let $a(x)$ be an atom in the sense of Definition 1.2 and suppose $\operatorname{supp} a \subset B$, $B = B(z,r)$, and $a = L^n b$ for some $b \in D(L^n)$, $\operatorname{supp} b \subset B$, and $\|b\|_\infty \le r^{2n}|B|^{-1/p}$.

Let $\varphi \in C_0^\infty(\mathbb{R})$ be real-valued and even with $\operatorname{supp}\varphi \subset [-1,1]$, $\varphi(0) = 1$, and $\varphi^{(\nu)}(0) = 0$ for $\nu \ge 1$. By Theorem 2.3 applied with $G(\lambda) = \varphi(\lambda)$ and $G(\lambda) = \lambda^{2n}\varphi(\lambda)$, it follows that $\varphi(t\sqrt{L})$ and $L^n\varphi(t\sqrt{L})$ are kernel operators with kernels satisfying the following inequalities for any $\sigma > 0$ and all

$x, y \in \widetilde{X}$:

$$|\varphi(t\sqrt{L})(x,y)| \le c_\sigma |B(x,t)|^{-1}(1+t^{-1}\rho(x,y))^{-\sigma}, \tag{4.73}$$

$$|[L^n\varphi(t\sqrt{L})](x,y)| \le c_\sigma t^{-2n}|B(x,t)|^{-1}(1+t^{-1}\rho(x,y))^{-\sigma}. \tag{4.74}$$

We choose σ so that $\sigma > d/p + 2d$.

We need to estimate $|\varphi(t\sqrt{L})a(x)|$. Observe first that using (2.2) we have

$$\begin{aligned} |\varphi(t\sqrt{L})a(x)| &\le \int_X \frac{|a(y)|}{|B(x,t)|(1+t^{-1}\rho(x,y))^\sigma}\, d\mu(y) \\ &\le c|B|^{-1/p}, \qquad x \in 2B \cap \widetilde{X}. \end{aligned} \tag{4.75}$$

To estimate $|\varphi(t\sqrt{L})a(x)|$ for $x \in \widetilde{X} \setminus 2B$ we consider two cases:

Case 1: $0 < t \le r$. Let $x \in \widetilde{X} \setminus 2B$ and $y \in B$. From (1.2) and (2.1) it readily follows that

$$\begin{aligned} |B| \le c_0\Big(\frac{r}{t}\Big)^d |B(z,t)| &\le c_0^2\Big(\frac{r}{t}\Big)^d\Big(1+\frac{\rho(x,z)}{t}\Big)^d|B(x,t)| \\ &\le c_0^2\Big(1+\frac{\rho(x,z)}{t}\Big)^{2d}|B(x,t)|, \end{aligned}$$

where we have used $\rho(x,z) \ge r$. Combining this with (4.73) and the obvious inequality $\rho(x,z) \le \rho(x,y) + \rho(y,z) \le 2\rho(x,y)$ we obtain

$$\begin{aligned} |\varphi(t\sqrt{L})(x,y)| &\le c_\sigma |B(x,t)|^{-1}(1+t^{-1}\rho(x,y))^{-\sigma} \\ &\le c|B|^{-1}(1+t^{-1}\rho(x,z))^{-\sigma+2d}. \end{aligned}$$

In turn, this leads to

$$\begin{aligned} |\varphi(t\sqrt{L})a(x)| &= \Big|\int_B \varphi(t\sqrt{L})(x,y)a(y)\, d\mu(y)\Big| \\ &\le \frac{c|B|^{-1-1/p}}{(1+t^{-1}\rho(x,z))^{\sigma-2d}}\int_B 1\, d\mu(y) = \frac{c|B|^{-1/p}}{(1+t^{-1}\rho(x,z))^{\sigma-2d}}. \end{aligned}$$

From this and (4.75) we infer

$$\begin{aligned} \|\varphi(t\sqrt{L})a\|_{L^p}^p &= \|\varphi(t\sqrt{L})a\|_{L^p(2B)}^p + \|\varphi(t\sqrt{L})a\|_{L^p(X\setminus 2B)}^p \\ &\le c\int_{2B}|B|^{-1}\, d\mu(x) + c\int_X \frac{|B|^{-1}\, d\mu(x)}{(1+t^{-1}\rho(x,z))^{(\sigma-2d)p}} \\ &\le c' + c|B|^{-1}|B(z,t)| \le c. \end{aligned} \tag{4.76}$$

Here we have used $(\sigma - 2d)p > d$ and (2.2).

Case 2: $t > r$. Let $x \in \widetilde{X} \setminus 2B$ and $y \in B$. Using (2.1) we obtain

$$|B| = |B(z,r)| \le |B(z,t)| \le c_0(1+\rho(x,z)/t)^d|B(x,t)|$$

and as before $\rho(x,z) \le 2\rho(x,y)$. These coupled with (4.74) lead to

$$|[L^n\varphi(t\sqrt{L})](x,y)| \le c(r/t)^{2n}|B|^{-1}(1+t^{-1}\rho(x,z))^{-\sigma+d}.$$

This and $\|b\|_\infty \le r^{2n}|B|^{-1/p}$ imply

$$\begin{aligned}|\varphi(t\sqrt{L})a(x)| &= \Big|\int_B L^n\varphi(t\sqrt{L})(x,y)b(y)\,d\mu(y)\Big|\\ &\le \frac{c(r/t)^{2n}|B|^{-1-1/p}}{(1+t^{-1}\rho(x,z))^{\sigma-2d}}\int_B 1\,d\mu(y) = \frac{c(r/t)^{2n}|B|^{-1/p}}{(1+t^{-1}\rho(x,z))^{\sigma-2d}}.\end{aligned}$$

We use this and (4.75) to obtain

$$\begin{aligned}\|\varphi(t\sqrt{L})a\|_{L^p}^p &= \|\varphi(t\sqrt{L})a\|_{L^p(2B)}^p + \|\varphi(t\sqrt{L})a\|_{L^p(X\setminus 2B)}^p\\ &\le c\int_{2B}|B|^{-1}\,d\mu(x) + c\int_X \frac{(r/t)^{2np}|B|^{-1}\,d\mu(x)}{(1+t^{-1}\rho(x,z))^{(\sigma-2d)p}}\\ &\le c' + c(r/t)^{2np}|B|^{-1}|B(z,t)| \le c' + cc_0(r/t)^{2np}(t/r)^d\\ &= c' + cc_0(r/t)^{2np-d} \le c.\end{aligned}$$

Here we have used the fact that $|B(z,t)| \le c_0(t/r)^d|B(z,r)|$ by (1.2) and that $n \ge d/(2p)$. In light of Theorem 3.7 the above and (4.76) yield (4.72). ∎

We are now prepared to complete the proof of the embedding $H_A^p \subset H^p$. Assume that $f \in \mathbb{H}_A^p$ (see Definition 1.3). Then there exist atoms $\{a_k\}_{k\ge1}$ and coefficients $\{\lambda_k\}_{k\ge1}$ such that $f = \sum_k \lambda_k a_k$ (convergence in L^2) and $\sum_k |\lambda_k|^p \le 2\|f\|_{\mathbb{H}_A^p}^p$.

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be real-valued and even, and $\varphi(0) = 1$. Then in light of Proposition 2.5,

$$\varphi(t\sqrt{L})f(x) = \sum_{k=1}^\infty \lambda_k\varphi(t\sqrt{L})a_k(x), \qquad x \in X,\ t > 0,$$

and hence

$$\sup_{t>0}|\varphi(t\sqrt{L})f(x)| \le \sum_{k=1}^\infty |\lambda_k|\sup_{t>0}|\varphi(t\sqrt{L})a_k(x)|,$$

which is the same as $M(f;\varphi)(x) \le \sum_{k=1}^\infty |\lambda_k|M(a_k;\varphi)(x)$. Therefore, for $0 < p \le 1$,

$$\|M(f;\varphi)\|_{L^p}^p \le \sum_{k=1}^\infty |\lambda_k|^p\|M(a_k;\varphi)\|_{L^p}^p \le c\sum_{k=1}^\infty |\lambda_k|^p \le c\|f\|_{\mathbb{H}_A^p}^p.$$

On account of Theorem 3.7 this implies $\|f\|_{H^p} \le c\|f\|_{H_A^p}$. ∎

4.3. Proof of Theorem 1.4 in the compact case. We proceed quite similarly to the noncompact case. Therefore, we shall only indicate the modifications that need to be made.

To prove the embedding $H^p \subset H^p_A$ assume $f \in H^p$, $0 < p \le 1$. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be just as in the proof in the noncompact case. Instead of (4.17) we use (4.16) to represent f, that is,

$$f = \varphi_j \tilde{\varphi}_j f + \sum_{k=j+1}^{\infty} \psi_k \tilde{\psi}_k f =: f_0 + f_1 \quad (\text{convergence in } L^2), \tag{4.77}$$

where j is the maximal integer such that $B(x_0, 2^{-j}) = X$, and φ_j, $\tilde{\varphi}_j$, ψ_k and $\tilde{\psi}_k$ are as in (4.3). For the decomposition of f_1 we just repeat the proof from §4.1. On the other hand, as in (4.7) we have $|\varphi_j \tilde{\varphi}_j f(x)| \le c\mathcal{M}_N(f)(y)$ for all $x, y \in \widetilde{X}$, and hence

$$\|\varphi_j \tilde{\varphi}_j f\|_\infty \le c|X|^{-1/p} \|\mathcal{M}_N(f)(y)\|_{L^p} \le c_* |X|^{-1/p} \|f\|_{H^p}.$$

We define the outstanding atom A (see (1.4)) by $A := c_*^{-1} \|f\|_{H^p}^{-1} \varphi_j \tilde{\varphi}_j f$ and set $\lambda_A := c_* \|f\|_{H^p}$. Clearly, $\|A\|_\infty \le |B|^{-1/p}$ and $\lambda_A A = \varphi_j \tilde{\varphi}_j f = f_0$. Thus we arrive at the claimed atomic decomposition of f.

The proof of the embedding $H^p_A \subset H^p$ runs in the footsteps of the proof in the noncompact case from §4.2. We only have to show in addition the estimate $\|A\|_{H^p} \le c < \infty$ for any outstanding atom A as in (1.4). But, this estimate follows readily from estimate (4.75) applied to A. ∎

5. Appendix

Proof of Lemma 4.4. Choose $\{B(\xi_j, \rho(\xi_j)/5)\}_{j\in\mathbb{N}}$ to be a maximal disjoint subcollection of $\{B(x, \rho(x)/5)\}_{x\in\Omega}$, whose existence follows by Zorn's lemma. Then (b) is obvious.

We now establish (a). Assume to the contrary that there exists $x \in \Omega$ such that $x \notin \bigcup_{j\in\mathbb{N}} B(\xi_j, \rho_j/2)$. From the construction of $\{B(\xi, \rho_j/5)\}_{j\in\mathbb{N}}$ it follows that $B(x, \rho(x)/5) \cap B(\xi, \rho_j/5) \ne \emptyset$ for some $j \in \mathbb{N}$. We claim that

$$\rho(\xi_j) > (2/3)\rho(x). \tag{5.1}$$

Indeed, assume that $\rho(\xi_j) \le (2/3)\rho(x)$. Then

$$\rho(x, \xi_j) < (1/5)(\rho(\xi_j) + \rho(x)) \le (1/3)\rho(x).$$

Therefore, $B(\xi_j, \rho_j) \subset B(x, \rho(x, \xi_j) + \rho(\xi_j)) \subset B(x, \rho(x))$, where the first inclusion is strict. This implies $B(\xi_j, (1+\eta)\rho_j) \subset B(x, \rho(x)) \subset \Omega$ for some $\eta > 0$. But from the definition of ρ_j it follows that $B(\xi_j, (1+\eta)\rho_j) \cap \Omega^c \ne \emptyset$. This is a contradiction, which proves (5.1). From (5.1) we infer

$$\rho(x, \xi_j) < (1/5)(\rho(\xi_j) + \rho(x)) \le (1/5)(1 + 3/2)\rho(\xi_j) = (1/2)\rho(\xi_j),$$

which verifies (a).

To prove (c) assume $B(\xi_j, 3\rho_j/4) \cap B(\xi_\nu, 3\rho_\nu/4) \neq \emptyset$ for some $j, \nu \in \mathbb{N}$. We shall show that $\rho_j \le 7\rho_\nu$. We proceed as above. Assume that $\rho_j > 7\rho_\nu$. Then $\rho(\xi_j, \xi_\nu) \le (3/4)(\rho_j + \rho_\nu) \le (6/7)\rho_j$ yielding

$$B(\xi_\nu, \rho_\nu) \subset B(\xi_j, \rho(\xi_j, \xi_\nu) + \rho_\nu) \subset B(\xi_j, (6/7)\rho_j + (1/7)\rho_j) = B(\xi_j, \rho_j),$$

where the first inclusion is strict. As above this leads to a contradiction, which shows that $\rho_j \le 7\rho_\nu$.

To prove (d), assume that the balls $B(\xi_{\nu_m}, 3\rho_{\nu_m}/4)$, $m = 1, \dots, K$, intersect $B(\xi_j, 3\rho_j/4)$. Then from the above, $\rho_j \le 7\rho_{\nu_m}$, $m = 1, \dots, K$. Using this, (2.1) and (1.2) we get

$$\begin{aligned} |B(\xi_j, 8\rho_j)| &\le c_0\left(1 + \frac{\rho(\xi_j, \xi_{\nu_m})}{8\rho_j}\right)^d |B(\xi_{\nu_m}, 8\rho_j)| \\ &\le c_0^2\left(1 + \frac{\rho(\xi_j, \xi_{\nu_m})}{8\rho_j}\right)^d 40^d |B(\xi_j, \rho_{\nu_m}/5)|. \end{aligned}$$

However, by (c), $\rho(\xi_j, \xi_{\nu_m}) \le (3/4)(\rho_j + \rho_{\nu_m}) \le 6\rho_j$. Therefore,

$$|B(\xi_j, 8\rho_j)| \le c_0^2 70^d |B(\xi_j, \rho_{\nu_m}/5)|,$$

and summing up we obtain

$$K|B(\xi_j, 8\rho_j)| \le 70^d c_0^2 \sum_{m=1}^{K} |B(\xi_j, \rho_{\nu_m}/5)|. \tag{5.2}$$

On the other hand, by (b) the balls $B(\xi_{\nu_m}, \rho_{\nu_m}/5)$, $m = 1, \dots, K$, are disjoint, and as each ball $B(\xi_{\nu_m}, 3\rho_{\nu_m}/4)$ intersects $B(\xi_j, 3\rho_j/4)$ and $\rho_{\nu_m} \le 7\rho_j$, we have

$$B(\xi_{\nu_m}, \rho_{\nu_m}/5) \subset B(\xi_j, 3\rho_j/4 + (3/4 + 1/5)\rho_{\nu_m}) \subset B(\xi_j, 8\rho_j).$$

Consequently, $\sum_{m=1}^{K} |B(\xi_{\nu_m}, \rho_{\nu_m}/5)| \le |B(\xi_j, 8\rho_j)|$. This coupled with (5.2) yields $K \le 70^d c_0^2$. ∎

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