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HIGH ORDER GEOMETRIC SMOOTHNESS FOR CONSERVATION LAWS*

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Abstract. The smoothness of the solutions of 1D scalar conservation laws is investigated and it is shown that if the initial value has smoothness of order α in L^q with $\alpha > 1$ and $q = 1/\alpha$, this smoothness is preserved at any time $t > 0$ for the graph of the solution viewed as a function in a suitably rotated coordinate system. The precise notion of smoothness is expressed in terms of a scale of Besov spaces which also characterizes the functions that are approximated at rate $N^{-\alpha}$ in the uniform norm by piecewise polynomials on N adaptive intervals. An important implication of this result is that a properly designed adaptive strategy should approximate the solution at the same rate $N^{-\alpha}$ in the Hausdorff distance between the graphs.

Keywords:

1. Introduction

Solutions to hyperbolic equations derived from nonlinear conservation laws

$$\partial_t u + \mathrm{Div}_x[f(u)] = 0, \quad u(x,0) = u_0(x), \tag{1.1}$$

may develop discontinuities even if the initial data is smooth. This well known state of fact is the source of both theoretical difficulties — classical solutions should be replaced by weak solutions and side conditions need to be appended in order to ensure their uniqueness — as well as numerical difficulties — conventional discretization schemes may fail to converge and their convergence rate is in all cases limited by the lack of smoothness of the solution. We refer the reader to [6, 7, 10] for a general introduction to conservation laws.

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In the case of scalar conservation laws, the classical theory developed by Kruzkov [8] ensures the uniqueness of an entropy solution $u(x,t)$. This solution is also stable in L^1, i.e.,

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \le \|u_0 - v_0\|_{L^1} \tag{1.2}$$

for two solutions u and v with initial data u_0 and v_0, and satisfies the BV diminishing property

$$\|u(\cdot,t)\|_{BV} \le \|u_0\|_{BV}. \tag{1.3}$$

The BV boundedness plays a pivotal role in proving the convergence of numerical methods and deriving convergence rates with respect to the mesh size. As already mentioned, these rates are inherently limited by the lack of smoothness: the approximation u_h of a function u by piecewise polynomials on a uniform mesh cannot converge in L^1 with a rate better than $\mathcal{O}(h)$ when u has an isolated jump.

Adaptive methods offer a better compromise between error and number of degrees of freedom, especially when the solution is piecewise smooth with isolated singularities. From approximation theory point of view these methods correspond to approximation from piecewise polynomials of a fixed degree on N intervals. Note that this is a nonlinear set since the N intervals may vary with the function being approximated and therefore this type of approximation is referred to as *nonlinear approximation*. A precise description of those functions which can be approximated in L^1 at rate $N^{-\alpha}$ by such piecewise polynomial functions is given by the Besov space $B^\alpha_{q,q}$ with $1/q = 1 + \alpha$, which consists of all functions $u \in L^q$ such that

$$|u|^q_{B^\alpha_{q,q}} := \int_0^\infty [t^{-\alpha}\omega_k(u,t)_q]^q dt/t < \infty, \tag{1.4}$$

where k is an integer strictly larger than α and $\omega_k(u,t)_q := \sup_{|h|\le t} \left\|\Delta_h^k u\right\|_{L^q}$ is the kth order L^q modulus of smoothness. The norm in $B^\alpha_{q,q}$ is defined by

$$\|u\|_{B^\alpha_{q,q}} := \|u\|_{L^q} + |u|_{B^\alpha_{q,q}}. \tag{1.5}$$

Roughly speaking, the functions in $B^\alpha_{q,q}$ have α derivatives in L^q. We refer to [2] as a general survey on nonlinear approximation.

In a series of papers [3, 4, 11], DeVore and Lucier have explored the smoothness properties of 1D scalar conservation laws using the above Besov spaces. They have shown that for all $\alpha > 0$, if the initial condition u_0 belongs to $B^\alpha_{q,q}$ with $1/q = 1 + \alpha$, then this property holds for the solution for all $t > 0$. The theorem of DeVore–Lucier shows that the solutions of conservation laws have an arbitrarily high order of smoothness $\alpha > 0$ whenever the smoothness is measured in L^q with $1/q = 1 + \alpha$, and therefore $q < 1$. From a numerical perspective, it also indicates that a properly designed adaptive strategy should approximate the solution in L^1 with an arbitrarily high rate of convergence with respect to the number of degrees of freedom. The proof of this theorem is based on the equivalence between smoothness and rate of nonlinear approximation, according to the following scheme:

1. The initial data $u_0 \in B^\alpha_{q,q}$ is approximated at rate $N^{-\alpha}$ by a piecewise polynomial function v_0 on N intervals.
2. Then by the L^1 stability (1.2) the solution u at time $t > 0$ is approximated at the same rate $N^{-\alpha}$ by the solution v with initial value v_0.
3. This rate of approximation allows to derive that $u \in B^\alpha_{q,q}$.

The main difficulty in this approach resides in the last step since it is no longer true that v is a piecewise polynomial on N intervals.

Since one of the goals of adaptive methods is to achieve uniformly accurate approximation, one could hope for similar results with the L^1 norm replaced by the uniform (L^∞) norm as a measure of the error. However, such results are impossible since there is no stability in the uniform norm due to the development of discontinuities. A natural alternative is to measure the closeness between solutions and approximate solutions in the *Hausdorff distance between their completed graphs*, i.e.,

$$d(u,v) = d_H(G_u, G_v),$$

where G_f denotes the completed graph of the function f and

$$d_H(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right\}$$

denotes the Hausdorff distance between the sets A and B (with $|\cdot|$ denoting the Euclidean distance in $\mathbb{R}^2$). Here the *completed graph* G_f of a function f is defined as the minimal closed set in $\mathbb{R}^2$ which contains the graph of f and is convex with respect to the y-direction, i.e., it is y-simple. It is easy to see that if $f \in BV$ and $f(x^-) \le f(x) \le f(x^+)$ for every x, then to obtain G_f one has to add to the graph of f every segment in the plane connecting the points $(x, f(x^-))$ and $(x, f(x^+))$ at every point x, where f is discontinuous (see [13]). The distance $d(u,v)$ is a natural substitute for the L^∞ distance for discontinuous functions for two reasons: on the one hand it measures the closeness in L^∞ in regions where one of the functions is smooth enough since one easily checks that

$$\|u-v\|_{L^\infty} \le d(u,v)[\|u'\|_{L^\infty} + 1]$$

and on the other hand it measures how accurately a sharp transition in u is matched in the x-direction by a sharp transition in v. In contrast to the L^∞ norm, stability results in the Hausdorff metric are available from [1], where it was recently proved that for 1D scalar conservation laws one has

$$d(u,v) \le C(t)d(u_0,v_0) \tag{1.6}$$

with $C(t) \sim 1 + t$.

In this paper, we shall use these results to establish high order smoothness results on the graph of the solution viewed as a function in a suitably rotated coordinate system. This approach is applicable in the case of strictly convex fluxes f, satisfying

$$0 < m \le f''(u). \tag{1.7}$$

In a case like this, we invoke the Oleinik inequality which ensures that the entropy solution u of (1.1) satisfies at time $t > 0$,

$$-\infty \le u' \le \frac{1}{mt}. \tag{1.8}$$

This inequality ensures that the graph of u is the graph of a Lipschitz function $\tilde{u}$ in a suitably rotated coordinate system (which will be precisely specified in Sec. 3). In such a coordinate system the L^∞ distance between two solutions is equivalent to the Hausdorff distance between their graphs in the original coordinate system. This fact is illustrated in Fig. 1.

We shall prove that the function $\tilde{u}$ can be approximated in L^∞ by piecewise polynomials on N intervals at rate $N^{-\alpha}$, whenever u_0 satisfies a similar property. As it will be explained in Sec. 2, the set of functions which can be approximated in the uniform norm at rate (roughly) $N^{-\alpha}$ with $\alpha > 1$ by such piecewise polynomials is given by the space

$$\tilde{B}^\alpha := \{u \in W^{1,1}(\mathbb{R}) : u' \in B^{\alpha-1}_{q,q},\ q = 1/\alpha\}. \tag{1.9}$$

The norm in $\tilde{B}^\alpha$ is defined by

$$\|u\|_{\tilde{B}^\alpha} := \|u\|_{L^\infty} + \|u'\|_{B^{\alpha-1}_{q,q}}. \tag{1.10}$$

Notice that this space is slightly smaller than the Besov space $B^\alpha_{q,q}$ which may contain discontinuous functions if $q < 1$.

We next state our main result.

Theorem 1.1. *Assume that u_0 is a compactly supported function which satisfies $u_0' \le M$. Then for all $\alpha > 1$ and time $t > 0$, the rotated solution $\tilde{u}$ satisfies*

$$\|\tilde{u}\|_{\tilde{B}^\alpha} \lesssim \|u_0\|_{\tilde{B}^\alpha} + 1, \tag{1.11}$$

where the constant in $\lesssim$ depends only on t and M.

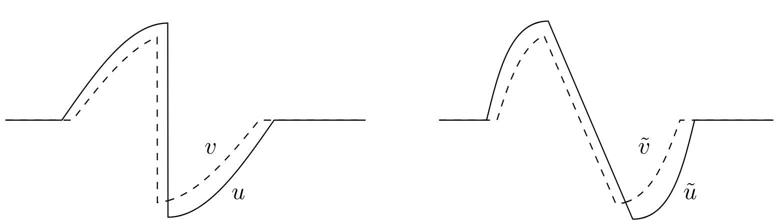

Fig. 1. Change of coordinate system.

From numerical perspective, this result indicates that a properly designed adaptive strategy should approximate the solution in the Hausdorff distance at an arbitrarily high rate with respect to the number of degrees of freedom.

The paper is organized as follows: In Sec. 2, we give some preliminary results for nonlinear approximation in L^∞ and on the Hausdorff stability of conservation laws. Using these results, we develop in Sec. 3 the strategy of DeVore–Lucier from [3, 4], namely, we construct approximate solutions which approximate the true solution at rate $N^{-\alpha}$ in the Hausdorff metric, and as a consequence in L^∞ with respect to the rotated coordinate system. The "return ticket" which allows to derive the smoothness of $\tilde{u}$ from the approximation rate relies on inverse estimates which are the objective of Sec. 4.

2. Preliminary Results

2.1. *Nonlinear piecewise polynomial approximation*

For a fixed compact interval I and a positive integer k, let us denote by Σ_n the set of all piecewise polynomials of degree not exceeding k with no more than 2^n pieces on I. Then for a given $u \in L^p(I)(0 < p \le \infty)$ the error of best L^p approximation to u from Σ_n is defined by

$$\sigma_n(u)_p := \inf_{S\in\Sigma_n} \|u - S\|_{L^p}. \tag{2.1}$$

If some S_n realizes this infimum, it is said to be a best L^p approximation to u from Σ_n. We find useful the notion of a *near-best* approximation, that corresponds to $\|u - S_n\|_{L^p} \le C\sigma_n(u)_p$ for some constant $C \ge 1$ independent of n and u.

In order to describe the approximation rate, it is convenient to introduce the approximation space $\mathcal{A}^\alpha_q(L^p)$, defined as the set of all functions $u \in L^p$ such that

$$\|u\|_{\mathcal{A}^\alpha_q(L^p)} := \left(\sum_{n=-1}^{\infty} [\, 2^{n\alpha}\sigma_n(u)_p \,]^q\right)^{1/q} \tag{2.2}$$

is finite. Here we use the convention $\Sigma_{-1} = \{0\}$, so that $\sigma_{-1}(u)_p := \|u\|_{L^p}$. Clearly $\mathcal{A}^\alpha_\infty(L^p)$ is the set of functions which are approximated in L^p by piecewise polynomials with accuracy $\mathcal{O}(2^{-n\alpha})$ and $\mathcal{A}^\alpha_q(L^p)$ is a slight variation of this set since $\mathcal{A}^{\alpha+\varepsilon}_\infty(L^p) \subset \mathcal{A}^\alpha_q(L^p) \subset \mathcal{A}^\alpha_\infty(L^p)$ for any $\varepsilon > 0$. We also recall that if $\sigma_n(u)_p \to 0$ as $n \to \infty$, one obtains an equivalent norm in $\mathcal{A}^\alpha_q(L^p)$ by replacing $\sigma_n(u)_p$ by $\|S_{n+1} - S_n\|_{L^p}$, where S_n is a near-best approximation to u from Σ_n. Indeed, clearly $\|S_{n+1} - S_n\|_{L^p} \lesssim \sigma_{n+1}(u)_p + \sigma_n(u)_p$ with a constant independent of n. On the other hand, S_n converges to u in L^p and hence $\|u - S_n\|_{L^p}$ can be bounded by $\sum_{n'\ge n} \|S_{n'+1} - S_{n'}\|_{L^p}$, and we complete the argument by the discrete Hardy inequality.

Since the work of DeVore and Popov [5], it is known that when $\alpha < k+1$, $\mathcal{A}^\alpha_q(L^1)$ coincides with the Besov space $B^\alpha_{q,q}$ with $1/q = 1 + \alpha$ and they have equivalent norms. In this paper, we are interested in piecewise polynomial approximation of continuous functions in the uniform norm. In this context, Σ_n is redefined as the set of all *continuous* piecewise polynomials of degree $\leq k$ with no more that 2^n polynomial pieces. This type of approximation is studied by Petrushev in [12], where the following Jackson and Bernstein estimates are established:

$$\sigma_n(u)_\infty \lesssim 2^{-\beta n}\|u'\|_{B^{\beta-1}_{r,r}} \tag{2.3}$$

and

$$u \in \Sigma_n \Rightarrow \|u'\|_{B^{\beta-1}_{r,r}} \lesssim 2^{\beta n}\|u\|_{L^\infty}, \tag{2.4}$$

with $1 < \beta < k+1$ and $r = 1/\beta$. These estimates are the classical vehicle for characterizing the approximation spaces $\mathcal{A}^\alpha_q(L^\infty)$ for $0 < \alpha < \beta$ in terms of the real interpolation spaces $(L^\infty, \tilde{B}^\beta)_{\frac{\alpha}{\beta},q}$, where

$$\tilde{B}^\beta := \big\{u : u' \in B^{\beta-1}_{r,r},\ r = 1/\beta\big\}. \tag{2.5}$$

In the following, we shall prove directly that $\mathcal{A}^\alpha_q(L^\infty)$ in fact coincides with $\tilde{B}^\alpha$ for $1 < \alpha < k+1$. As already mentioned, $\tilde{B}^\alpha$ is slightly smaller than $B^\alpha_{q,q}$ and does not contain discontinuous functions.

Lemma 2.1. *We have* $\mathcal{A}^\alpha_q(L^\infty) = \tilde{B}^\alpha$, $q = 1/\alpha$, *with equivalent norms.*

Proof. Assume that $u \in \mathcal{A}^\alpha_q(L^\infty)$ and denote by S_n $(n \geq 0)$ a near-best L^∞ approximation to u from Σ_n. We consider the discontinuous piecewise polynomial $T_n := S'_n$ of degree $k-1$ as an approximation to u'. Note that any polynomial S of degree k satisfies

$$\|S'\|_{L^1([a,b])} \leq C\|S\|_{L^\infty([a,b])},$$

where the constant C depends on k, but is independent of the interval $[a,b]$ by a scaling argument. Since $T_n - T_{n-1}$ is a piecewise polynomial on at most $\frac{3}{2}2^n$ intervals I_j, we have

$$\|T_n - T_{n-1}\|_{L^1} \leq \sum_j \|T_n - T_{n-1}\|_{L^1(I_j)} \lesssim 2^n\|S_n - S_{n-1}\|_{L^\infty}.$$

This gives

$$\sum_{n=-1}^{\infty} [\,2^{n(\alpha-1)}\|T_n - T_{n-1}\|_{L^1}\,]^q \lesssim \|u\|^q_{\mathcal{A}^\alpha_q(L^\infty)},$$

which in turn shows that T_n converges to an L^1 function which is necessarily u'. It follows that

$$\|u'\|_{\mathcal{A}^{\alpha-1}_q(L^1)} \lesssim \|u\|_{\mathcal{A}^\alpha_q(L^\infty)}$$

and therefore, according to the result of [5] for piecewise polynomial approximation in L^1,

$$\|u'\|_{B^{\alpha-1}_{q,q}(L^1)} \lesssim \|u\|_{\mathcal{A}^\alpha_q(L^\infty)}.$$

Now since $\|u\|_{L^\infty} \le \|u\|_{\mathcal{A}_q^\alpha(L^\infty)}$, then

$$\|u\|_{\tilde{B}^\alpha} \lesssim \|u\|_{\mathcal{A}_q^\alpha(L^\infty)}.$$

For the estimate in the other direction, let us assume that $u \in \tilde{B}^\alpha$. Then $u' \in B_{q,q}^{\alpha-1}$ with $1/q = 1 + (\alpha - 1)$, and due to the result of [5] for piecewise polynomial approximation in L^1, there exists a sequence $(T_n)_{n\ge -1}$ of piecewise polynomials of degree $k-1$ with $T_{-1} = 0$ such that T_n converges to u' in L^1 and

$$\sum_{n=-1}^{\infty} 2^{(\alpha-1)qn} \|u' - T_n\|_{L^1}^q \lesssim \|u'\|_{B_{q,q}^{\alpha-1}}^q.$$

Clearly, there is a subdivision with at most 2^{n+1} intervals I_j such that T_n is a polynomial on each of them and

$$\|u' - T_n\|_{L^1(I_j)} \le 2^{-n} \|u' - T_n\|_{L^1}.$$

On each interval $I_j = [a_j, b_j]$, we define

$$P_{n+1}(x) := u(a_j) + \int_{a_j}^{x} T_n(s) ds \tag{2.6}$$

and further modify P_{n+1} into

$$S_{n+1}(x) := P_{n+1}(x) + (u(b_j) - P_{n+1}(b_j)) \frac{x - a_j}{b_j - a_j}. \tag{2.7}$$

Thus the resulting S_{n+1} is in Σ_{n+1}. On each I_j, we clearly have

$$|u(x) - P_{n+1}(x)| \le \|u' - T_n\|_{L^1(I_j)} \le 2^{-n} \|u' - T_n\|_{L^1}$$

and hence

$$|u(b_j) - P_{n+1}(b_j)| \frac{x - a_j}{b_j - a_j} \le 2^{-n} \|u' - T_n\|_{L^1}.$$

Consequently,

$$\|u - S_{n+1}\|_{L^\infty} \le 2^{-n+1} \|u' - T_n\|_{L^1},$$

which implies

$$\|u\|_{\mathcal{A}_q^\alpha(L^\infty)}^q \lesssim \|u\|_{L^\infty}^q + \|u'\|_{\mathcal{A}_q^{\alpha-1}(L^1)}^q.$$

Now invoking the result of [5] for piecewise polynomial approximation in L^1, we conclude

$$\|u\|_{\mathcal{A}_q^\alpha(L^\infty)} \lesssim \|u\|_{\tilde{B}^\alpha}.$$

The proof is complete. □

In the second part of the proof of Lemma 2.1, we constructed the approximation S_{n+1} to u by using that T_n approximates u' (see (2.6)–(2.7)). For future use, it will be useful to construct S_n so that if $u' \le M$, then S_n also satisfies $S_n' \le M$. To this end, we slightly modify the above construction as is described in the following. Once the intervals I_j are determined, we define on each of them a new approximation R_{n+1} to u' as the orthogonal projection of u' onto the polynomials of degree $k-1$, namely, R_{n+1} is defined on each I_j so that

$$\int_{I_j} [R_{n+1}(x) - u'(x)]x^\nu dx = 0, \quad \nu = 0, \dots, k-1.$$

Since this orthogonal projection is a near-best L^1 approximation, we have

$$\|u' - R_{n+1}\|_{L^1(I_j)} \lesssim \|u' - T_n\|_{L^1(I_j)}.$$

Inside I_j, there are at most $[k/2+1]$ disjoint intervals on which $R_{n+1}(x) > M$. On each of them we replace R_{n+1} by the constant M and on the remaining part $\tilde{I}_j$ of I_j we modify R_{n+1} as $M - c(M - R_{n+1})$, where c ensures that the integral of R_{n+1} on I_j remains unchanged. Note that since this integral is

$$\int_{I_j} R_{n+1} = \int_{I_j} u' \le M|I_j|,$$

then the constant

$$c := \frac{\int_{I_j} [M - R_{n+1}]}{\int_{\tilde{I}_j} [M - R_{n+1}]}$$

is necessarily in $[0,1]$ and consequently $M - c(M - R_{n+1}) \le M$ on $\tilde{I}_j$. The resulting function U_{n+a} has at most 2^{n+a} pieces with $a = 1 + [\log_2 k]$ and satisfies $U_{n+a} \le M$ everywhere. We finally remark that this modification can only improve the L^1 approximation error on I_j. Indeed, on the one hand

$$\|u' - U_{n+a}\|_{L^1(I_j \setminus \tilde{I}_j)} \le \|u' - R_{n+1}\|_{L^1(I_j \setminus \tilde{I}_j)} - \int_{I_j \setminus \tilde{I}_j} [R_{n+1} - M]$$

and on the other hand

$$\begin{aligned}
\|u' - U_{n+a}\|_{L^1(\tilde{I}_j)} &= \|u' - M - c(R_{n+1} - M)\|_{L^1(\tilde{I}_j)} \\
&\le \|u' - R_{n+1}\|_{L^1(\tilde{I}_j)} + (1-c)\|M - R_{n+1}\|_{L^1(\tilde{I}_j)} \\
&= \|u' - R_{n+1}\|_{L^1(\tilde{I}_j)} + \left(\int_{\tilde{I}_j} [M - R_{n+1}] - \int_{I_j} [M - R_{n+1}] \right) \\
&= \|u' - R_{n+1}\|_{L^1(\tilde{I}_j)} + \int_{I_j \setminus \tilde{I}_j} [R_{n+1} - M].
\end{aligned}$$

Therefore

$$\|u' - U_{n+a}\|_{L^1(I_j)} \le \|u' - R_{n+1}\|_{L^1(I_j)} \lesssim \|u' - T_n\|_{L^1(I_j)}. \tag{2.8}$$

We now define $S_{n+a} \in \Sigma_{n+a}$ on each interval I_j by

$$S_{n+a}(x) := u(a_j) + \int_{a_j}^x U_{n+a}(s) ds. \tag{2.9}$$

The continuity of S_{n+a} is ensured since by construction $\int_{I_j} U_{n+a} = \int_{I_j} u'$ and we clearly have $S'_{n+a} \le M$. We complete the argument as in the proof of Lemma 2.1, namely, we have

$$\|u - S_{n+a}\|_{L^\infty} \lesssim 2^{-n}\|u' - T_n\|_{L^1}$$

and hence

$$\|u\|_{\mathcal{A}_q^\alpha(L^\infty)} \le \left(\sum_{n=-1}^{\infty} [2^{n\alpha}\|u - S_n\|_{L^\infty}]^q\right)^{1/q} \lesssim \|u\|_{\tilde{B}^\alpha}, \tag{2.10}$$

where $S_n := 0$ for $-1 \le n < a$.

2.2. *Hausdorff stability and rotated graphs*

In [1], it was proved that scalar conservation laws are stable in the Hausdorff metric $d(\cdot,\cdot)$ with respect to perturbations of the initial condition. More precisely, if u and v are solutions of (1.1) with initial values u_0 and v_0, and if for some $M > 0$ the initial condition u_0 satisfies

$$u_0' \le M \quad \text{or} \quad u_0' \ge -M, \tag{2.11}$$

then we have

$$d(u, v) \le C(t)d(u_0, v_0), \quad t > 0, \tag{2.12}$$

with $C(t) \sim 1 + M(1 + t)$. A stability result is also established with respect to a perturbation of the flux function: If u and v are solutions of (1.1) with initial value u_0 and fluxes f and g, respectively, then at time $t > 0$, we have

$$d(u, v) \le C(t)\|f' - g'\|_{L^\infty} \tag{2.13}$$

with $C(t) \sim 1 + t$. These two results can be combined, namely, if u and v are solutions of (1.1) with initial value u_0 and v_0 and fluxes f and g, and if u_0 satisfies (2.11), then

$$d(u, v) \le C(t)[d(u_0, v_0) + \|f' - g'\|_{L^\infty}] \tag{2.14}$$

with $C(t) \sim 1 + M(1 + t)$.

As already explained in the introduction, our main idea is to employ the Oleinik inequality (1.8) to replace the Hausdorff distance by the L^∞ distance in a suitably rotated coordinate system. Indeed, assuming that u satisfies (1.8), it is readily seen that the graph of u is also the graph of a Lipschitz function $\bar{u}$ in the rotated coordinate system defined by

$$\begin{cases} \bar{x} = cx - sy \\ \bar{y} = sx + cy \end{cases} \tag{2.15}$$

with $c := \cos\theta$, $s := \sin\theta$, $\theta \in [0, \pi/2[$ such that

$$\tau := s/c = \tan\theta = mt/2. \tag{2.16}$$

One can indeed readily check that

$$-\tau^{-1} \le \bar{u}'(\bar{x}) \le 2\tau + \tau^{-1}. \tag{2.17}$$

Clearly, the rotated solution $\bar{u}$ is not compactly supported since it coincides with the function $\bar{y} = \tau\bar{x}$ outside the region corresponding to the support of u. In order to preserve the compactness of the support, we modify $\bar{u}$ by setting

$$\tilde{u} := \bar{u} - \tau\bar{x}. \tag{2.18}$$

Thus the new coordinate system is

$$\begin{cases} \tilde{x} = \bar{x} = cx - sy \\ \tilde{y} = c^{-1}y. \end{cases} \tag{2.19}$$

If u is supported on $I(t) = [a(t), b(t)]$, then $\tilde{u}$ is supported on $\tilde{I}(t) = [ca(t), cb(t)]$. Clearly, we still have a Lipschitz bound

$$|\tilde{u}(\tilde{x}) - \tilde{u}(\tilde{y})| \le \nu|\tilde{x} - \tilde{y}| \tag{2.20}$$

with

$$\nu := \tau + \tau^{-1}. \tag{2.21}$$

We also remark that if $u \in BV$, then $\tilde{u} \in BV$, and

$$|\tilde{u}|_{BV(\tilde{I})} \le c^{-1}\,|u|_{BV(I)} \tag{2.22}$$

which follows immediately from the definition of the total variation:

$$|u|_{BV} := \sup \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})|,$$

where the supremum is taken over all selections of points $x_0 < \cdots < x_n$ in the support of u.

It is easy to see that if $\tilde{u}$ and $\tilde{v}$ are obtained from u and v by such a change of the coordinate system, then

$$\|\tilde{u} - \tilde{v}\|_{L^\infty} = \|\bar{u} - \bar{v}\|_{L^\infty} \le (1+\nu)d(\bar{u}, \bar{v}) = (1+\nu)d(u, v)$$

and in the other direction,

$$d(u, v) = d(\bar{u}, \bar{v}) \le \|\bar{u} - \bar{v}\|_{L^\infty} = \|\tilde{u} - \tilde{v}\|_{L^\infty}.$$

Therefore, the Hausdorff distance between two solutions is equivalent to the L^∞ distance between the rotated solutions. In particular, if u and v are solutions of

(1.1) with initial values u_0 and v_0 and fluxes f and g, and if u_0 satisfies (2.11), then we have

$$\|\tilde{u} - \tilde{v}\|_{L^\infty} \le C(t)[\|u_0 - v_0\|_{L^\infty} + \|f' - g'\|_{L^\infty}] \tag{2.23}$$

with $C(t) \sim \nu[1 + M(1 + t)]$.

3. Proof of the Regularity Theorem

The proof of Theorem 1.1 relies on an approximation procedure by piecewise algebraic functions which stay close to the solution u in the Hausdorff metric for all $t > 0$. As shown above, this stability will hold in L^∞ in the coordinate system (2.19).

3.1. *Approximate solutions*

Assuming that $u_0 \in \tilde{B}^\alpha$ satisfies $u_0' \le M$, let S_n be the L^∞ approximation to u_0 defined in (2.9). We recall that S_n is made up of at most 2^n polynomial pieces of degree $\le k$ with $k > \alpha - 1$ and that it satisfies

$$S_n' \le M. \tag{3.1}$$

We also observe that since $S_n' = U_n$ is a near-best L^1 approximation of u_0', then

$$\|S_n'\|_{L^1} \le C\|u_0'\|_{L^1}$$

for some constant C and therefore

$$\|S_n\|_{BV} \le C\|u_0\|_{BV}. \tag{3.2}$$

Notice that S_n is not necessarily a near-best L^∞ approximation to u_0. However, (2.10) guarantees that it is good enough for our purposes. Clearly, there is an interval Ω (whose size may depend on $\|u_0\|_{BV}$) such that $u_0(x)$ and $S_n(x)$ belong to Ω for any x.

We next approximate the flux function. Assume that $f \in C^2$ and f is strictly convex, so that there exist two constants m and $\bar{m}$ such that

$$0 < m \le f'' \le \bar{m} \quad \text{on } \Omega.$$

We also assume that f belongs to $W^{r+1,\infty}(\Omega)$. Then by a classical spline approximation result, there exists an $r - 1$ times continuously differentiable piecewise polynomial function g_n of degree $\le r$ with uniform knots at the points $j2^{-n}$, $j \in \mathbb{Z}$, such that

$$\big\|f^{(l)} - g_n^{(l)}\big\|_{L^\infty(\Omega)} \le C2^{-n(r+1-l)}\|f^{(r+1)}\|_{L^\infty(\Omega)} \quad \text{for } l = 0, \dots, r. \tag{3.3}$$

Changing slightly the constants m and $\bar{m}$, we may assume that the functions g_n also satisfy

$$0 < m \le g_n'' \le \bar{m} \quad \text{on } \Omega. \tag{3.4}$$

We now define s_n as the entropy solution at time t of (1.1) with initial value S_n and flux g_n, and denote it by $\tilde{s}_n$ in the coordinate system (2.19). Before going any further, we observe that our stability result (2.23) combined with (3.3) guarantee that

$$\|\tilde{u} - \tilde{s}_n\|_{L^\infty} \le C(t)[\|u_0 - S_n\|_{L^\infty} + 2^{-nr}] \tag{3.5}$$

as well as

$$\|\tilde{s}_{n+1} - \tilde{s}_n\|_{L^\infty} \le C(t)[\|S_{n+1} - S_n\|_{L^\infty} + 2^{-nr}]. \tag{3.6}$$

Therefore, $\tilde{s}_n$ approximates $\tilde{u}$ with the same rate as S_n approximates u_0, up to an additional term 2^{-nr}. In the following, we assume that $\alpha + 1 < r$. In particular, we can set $r := k + 2$.

3.2. *Structure of the approximate solutions*

We recall that a function $y := y(x)$ is said to be *algebraic* on an interval J if there exists a polynomial F in two variables such that $F(x, y(x)) = 0$ for $x \in J$. We shall now describe the structure of the approximate solutions $\tilde{s}_n$ in terms of particular algebraic pieces (y, J).

Lemma 3.1. *There exists a partition of the support of $\tilde{s}_n$ into $\mathcal{O}(2^n)$ intervals such that on each interval J, the function $\tilde{s}_n$ coincides with an algebraic piece (y, J) of one of the following two types*:

Type I: *y satisfies $\|y'\|_{L^\infty(J)} \le \nu$ and the algebraic equation*

$$R(T(x)) = y(x) + \nu x, \quad x \in J, \tag{3.7}$$

where ν is defined in (2.21), *$T(x) := y(x) + \nu x - Q(y(x))$, and R and Q are algebraic polynomials of degrees $k(r-1)$ and $r-1$, satisfying*

$$\begin{array}{lll} (\mathbf{A_1}) & 2 \le Q' \le c_1 & \text{on } y(J), \\ (\mathbf{A_2}) & 0 < R' \le c_2 & \text{on } T(J), \end{array}$$

for two constants c_1 and c_2.

Type II: *y satisfies*

$$y(0) = y(x) + \nu x, \quad x \in J, \tag{3.8}$$

i.e., $\tilde{s}_n$ is affine on J with slope $-\nu$.

Proof. Following DeVore–Lucier [4], we begin by introducing two special types of points. First, let $\{a_i\}_{0 \le i \le A}$ denote the knots of S_n, that is, the points where S_n changes from one polynomial piece to another. By construction, $A \le 2^n$. Then let $\{b_i\}_{0 \le i \le B}$ denote the *isolated* points such that $S_n(b_i)$ is a knot of g_n, that is, $S_n(b_i) = j2^{-n}$ for some j. To count them, we shall denote by $\{\tilde{b}_i\}_{0 \le i \le \tilde{B}}$ all b_j's such that $S_n(b_{j-1}) = S_n(b_j)$ and we denote the remaining ones by $\{\bar{b}_i\}_{0 \le i \le \bar{B}}$. Now, we have $\mathrm{Var}_{[\bar{b}_i,\, \bar{b}_{i+1}]}(S_n) \ge 2^{-n}$ for each i, hence $\|S_n\|_{BV} \geq \sum_{i=0}^{\bar{B}-1} \mathrm{Var}_{[\bar{b}_i,\bar{b}_{i+1}]}(S_n) \geq \bar{B}\, 2^{-n}$ and we infer from (3.2) that $\bar{B} \lesssim \|u_0\|_{BV} 2^n$. On the other hand, if I_j is an interval where S_n coincides with the polynomial P_j, P_j' should vanish at least once in each $[\tilde{b}_i, \tilde{b}_{i+1}] \subset I_j$. Since P_j' is of degree not exceeding k and by definition there are no second type points in I_j when P_j is a constant, we see that $\tilde{B}$ is of order $\mathcal{O}(2^n)$, and so is B.

In [9], Lax shows that if the initial data S_n is continuous and the flux function g_n is strictly convex, the entropy solution s_n of (1.1) satisfies

$$s_n(x,t) = S_n(z), \quad \text{where } z := z(x,t) \text{ is a solution of } \frac{x-z}{t} = g_n'(S_n(z)).$$

There may be many solutions of this equation, but a minimization property picks a specific value $z(x,t)$. Lax shows that $z(x,t)$ is an increasing function of x for a fixed t. Shocks occur wherever $z(x,t)$ is discontinuous in x. If we denote by σ_i the positions of these shocks and set $z_i^- := z(\sigma_i^-, t)$ and $z_i^+ := z(\sigma_i^+, t)$, this means that the function

$$\mathcal{S}\colon z \;\to\; z + t\, g_n'(S_n(z)) \tag{3.9}$$

is increasing on each interval $[z_i^+, z_{i+1}^-]$, while $\mathcal{S}(z_i^-) = \mathcal{S}(z_i^+) = \sigma_i$. From our previous discussion, we can describe $\mathcal{S}$ as $\mathcal{O}(2^n)$ polynomial pieces of degree at most $k(r-2)$, so it follows that there cannot be more than $\mathcal{O}(2^n)$ shocks. In addition, we see that there is a partition $\{I_i^0\}_{1\leq i\leq C\,2^n}$ such that $\mathcal{S}$ is an increasing polynomial on each interval I_i^0 and satisfies

$$s_n(\mathcal{S}(z)) = S_n(z), \quad z \in I_i^0 \tag{3.10}$$

(here s_n is multivalued at the shocks), while the intervals $I_i^t := \mathcal{S}(I_i^0)$ recover $\mathbb{R}$ and overlap only at the boundaries. Writing $x = \mathcal{S}(z)$, this leads to

$$\mathcal{S}(x - t\, g_n'(s_n(x))) = x, \quad x \in I_i^t. \tag{3.11}$$

Finally, we observe that in the coordinate system (2.19), each algebraic piece (s_n, I_i^t) becomes a piece of Type I, while the shocks become pieces of Type II, as is seen from Fig. 1. Indeed, let us fix i and let P and Q denote the polynomials coinciding with $c^{-1}S_n(s\,\cdot)$ and $s^{-1}tg_n'(c\,\cdot)$ on $s^{-1}I_i^0$ and $c^{-1}S_n(I_i^0)$ respectively. Define also $R := Id + Q \circ P$ the polynomial which coincides with $s^{-1}\mathcal{S}(s\,\cdot)$ on $s^{-1}I_i^0$. After a little algebra, in the new coordinate system, (3.11) becomes

$$R(\tilde{s}_n(\tilde{x}) + \nu\tilde{x} - Q(\tilde{s}_n(\tilde{x}))) = \tilde{s}_n(\tilde{x}) + \nu\tilde{x},$$

which gives (3.7) with $J := \tilde{I}_i^t$. Then

$$Q' = \frac{t}{\tau}\, g_n''(c\,\cdot) \quad \text{and} \quad R' = 1 + t\, g_n''(S_n(s\,\cdot))S_n'(s\,\cdot),$$

and hence $(\mathbf{A_1})$–$(\mathbf{A_2})$ follow readily from (3.4) and (3.1) with $c_1 = 2\bar{m}/m$ and $c_2 = 1 + t\bar{m}M$. □

3.3. *An inverse estimate*

According to Lemma 3.1, each difference $\tilde{s}_n - \tilde{s}_{n-1}$ is made of $\mathcal{O}(2^n)$ algebraic pieces (A, J) which are differences of pieces of first or second type. Following DeVore and Lucier [4, Lemma 4.2], we can further split these pieces in order to obtain a partition consisting of $\mathcal{O}(2^n)$ pieces (A, J), each of them monotone together with all its derivatives of order $\le k+1$. We next state an inverse estimate for such pieces which will allow to complete the proof of Theorem 1.1.

Lemma 3.2. *If (A, J) is an algebraic piece of $\tilde{s}_n - \tilde{s}_{n-1}$, then*

$$\|A' \cdot \mathbb{1}_J\|_{B^{\alpha-1}_{q,q}} \lesssim \|A\|_{L^\infty(J)} + 2^{-(r-1)n} \tag{3.12}$$

with a constant independent of n.

This inverse estimate has a delicate proof which will be given in Sec. 4.

From (3.12), we next deduce an inverse inequality for the functions $\tilde{s}_n - \tilde{s}_{n-1}$. Assuming that $\{(A_i, J_i)\}_{1\le i\le C\,2^n}$ is a subdivision of $\tilde{s}_n - \tilde{s}_{n-1}$ into algebraic pieces, we observe that the continuity of each $\tilde{s}_n$ yields

$$\tilde{s}'_n - \tilde{s}'_{n-1} = \sum_{i=1}^{C2^n} A'_i \cdot \mathbb{1}_{J_i}.$$

Therefore, using the q-triangle inequality for $B^{\alpha-1}_{q,q}$, we have

$$\begin{aligned} \|\tilde{s}'_n - \tilde{s}'_{n-1}\|^q_{B^{\alpha-1}_{q,q}} &\le \sum_{i=1}^{C2^n} \|A'_i \mathbb{1}_{J_i}\|^q_{B^{\alpha-1}_{q,q}} \\ &\lesssim \sum_{i=1}^{C2^n} [\|A_i\|_{L^\infty(J_i)} + 2^{-n(r-1)}]^q \\ &\lesssim 2^n \|\tilde{s}_n - \tilde{s}_{n-1}\|^q_{L^\infty} + 2^{-n((r-1)q-1)} \end{aligned} \tag{3.13}$$

and using (3.6), it follows that

$$\|\tilde{s}'_n - \tilde{s}'_{n-1}\|^q_{B^{\alpha-1}_{q,q}} \lesssim 2^n \|S_{n+1} - S_n\|^q_{L^\infty} + 2^{-n((r-1)q-1)}.$$

From (3.5), it also appears that $\tilde{u}$ can be decomposed into a telescopic sum

$$\tilde{u} = \sum_{n=0}^{\infty} \tilde{s}_n - \tilde{s}_{n-1}.$$

Then applying again the q-triangle inequality, we obtain

$$\begin{aligned} \|\tilde{u}'\|^q_{B^{\alpha-1}_{q,q}} &\le \sum_{n=0}^{\infty} \|\tilde{s}'_n - \tilde{s}'_{n-1}\|^q_{B^{\alpha-1}_{q,q}} \\ &\lesssim \sum_{n=0}^{\infty} \left[2^n \|S_n - S_{n-1}\|^q_{L^\infty} + 2^{-n((r-1)q-1)}\right] \\ &\lesssim \|u_0\|^q_{\tilde{B}^\alpha} + 1, \end{aligned}$$

where we used our assumption $r - 1 > \alpha = 1/q$. The proof of Theorem 1.1 is thus complete except for the proof of Lemma 3.2.

4. Proof of the Inverse Estimate

In this section, n is a fixed positive integer and (A, J) denotes an algebraic piece of $\tilde{s}_n - \tilde{s}_{n-1}$.

4.1. *An intermediate estimate*

In order to prove Lemma 3.2, we first establish the following intermediate inverse inequality.

Lemma 4.1. *If (A, J) is an algebraic piece of $\tilde{s}_n - \tilde{s}_{n-1}$, then*

$$\|A'\|_{L^\infty(J)} \lesssim |J|^{-1}\big(\|A\|_{L^\infty(J)} + 2^{-(r-1)n}\big) \tag{4.1}$$

with a constant independent of n.

Proof. Let $y(x)$ and $\bar{y}(x)$ denote the algebraic pieces of $\tilde{s}_n$ and $\tilde{s}_{n-1}$ on the interval J. Several cases are possible, depending on whether y and $\bar{y}$ are of Type I or Type II. However, we observe that there is nothing to prove when y and $\bar{y}$ are both of Type II. Thus we can always assume that y is of Type I and set

$$\Theta(x) := 1 - R'(T)(1 - Q'(y)).$$

We begin by establishing the equivalences

$$|\Theta(x)| \sim 1, \quad x \in J, \tag{4.2}$$

and

$$|T(J)| \sim |J| \tag{4.3}$$

with constants of equivalence independent of n.

For the proof of (4.2), we first see using $(\mathbf{A_1})$–$(\mathbf{A_2})$ that $\|\Theta\|_{L^\infty(J)} \leq 1 + c_2(1 + c_1)$. In the other direction, differentiating both sides of (3.7) and the expression for $T(x)$ with respect to x yields

$$R'(T)T'(x) = y'(x) + \nu \tag{4.4}$$

and

$$T'(x) = \nu - y'(x)[Q'(y) - 1]. \tag{4.5}$$

Hence

$$y'(x)\Theta(x) = \nu[R'(T) - 1]. \tag{4.6}$$

Let $J_+ := \{x \in J; |1-R'(T)| \geq 1/2\}$ and $J_- := J \backslash J_+$. If $x \in J_+$, then $|y'(x)\Theta(x)| \geq \nu/2$, and using $|y'(x)| \leq \nu$ it follows that $|\Theta(x)| \geq 1/2$. In the case when $x \in J_-$, we infer from the positivity of $R'(T)$ on J that $1/2 > 1 - R'(T)$, and using $(\mathbf{A_1})$, it follows that

$$\begin{aligned} |\Theta(x)| &\geq R'(T)Q'(y) - |1 - R'(T)| \\ &\geq (1/2)Q'(y) - 1/2 \\ &\geq 1/2. \end{aligned}$$

Hence $|\Theta(x)| \geq 1/2$ for $x \in J$ and the proof of (4.2) is complete.

We turn to the proof of (4.3). From (4.5), it is clear that $\|T'\|_{L^\infty(J)} \leq \nu(2 + c_1)$. To bound $T'(x)$ from below, suppose first that $y'(x) \geq 0$. Then (4.4) together with $(\mathbf{A_2})$ yields $T'(x) \geq \nu/c_2$. If $y'(x) \leq 0$, then (4.5) along with $(\mathbf{A_1})$ implies $T'(x) \geq \nu$, and (4.3) follows.

We recall the following classical inequalities, valid for arbitrary intervals G, G' such that $G \subset G'$, and a polynomial P of degree $\leq l$:

$$(\mathbf{P_1}) \quad \|\mathbf{P}\|_{\mathbf{L^\infty(G')}} \leq C \left(\frac{|G'|}{|G|}\right)^l \|P\|_{L^\infty(G)},$$

$$(\mathbf{P_2}) \quad \|\mathbf{P'}\|_{\mathbf{L^\infty(G)}} \leq C|G|^{-1}\|P\|_{L^\infty(G)}.$$

We now consider the case where $\bar{y}$ is of Type II. By (4.6), (4.2) and (3.8), we have

$$\begin{aligned} \|y' - \bar{y}'\|_{L^\infty(J)} &= \nu\|\Theta^{-1}(R'(T) - 1) + 1\|_{L^\infty(J)} \\ &= \nu\|\Theta^{-1}R'(T)Q'(y)\|_{L^\infty(J)} \\ &\lesssim \|R'(T)\|_{L^\infty(J)} \\ &\lesssim \|R'\|_{L^\infty(T(J))} \\ &\lesssim |T(J)|^{-1}\|R - \bar{y}(0)\|_{L^\infty(T(J))} \\ &\lesssim |J|^{-1}\|R(T) - \bar{y}(0)\|_{L^\infty(J)} \\ &\lesssim |J|^{-1}\|y - \bar{y}\|_{L^\infty(J)}, \end{aligned}$$

which proves the lemma in this case. Here the first inequality is again (4.2) together with $(\mathbf{A_1})$, the third one is $(\mathbf{P_2})$, the fourth one is (4.3), and the last one is (3.7) together with (3.8).

Let now y and $\bar{y}$ be both of Type I. We use (4.6), (4.2) and $(\mathbf{A_2})$ to obtain

$$\begin{aligned} \|y' - \bar{y}'\|_{L^\infty(J)} &= \nu \,\|(\Theta\,\bar{\Theta})^{-1}[\bar{\Theta}(R'(T) - 1) - \Theta(\bar{R}'(\bar{T}) - 1)]\|_{L^\infty(J)} \\ &\lesssim \quad \|\bar{\Theta}(R'(T) - 1) - \Theta(\bar{R}'(\bar{T}) - 1)\|_{L^\infty(J)} \\ &\lesssim \quad \|R'(T) - \bar{R}'(\bar{T})\|_{L^\infty(J)} + \|\Theta - \bar{\Theta}\|_{L^\infty(J)}. \end{aligned} \tag{4.7}$$

Therefore, the lemma will follow if we establish the estimates:

$$\|R'(T) - \bar{R}'(\bar{T})\|_{L^\infty(J)} \lesssim |J|^{-1}[\|y - \bar{y}\|_{L^\infty(J)} + 2^{-rn}] \tag{4.8}$$

and

$$\|\Theta - \bar{\Theta}\|_{L^\infty(J)} \lesssim |J|^{-1}[\|y - \bar{y}\|_{L^\infty(J)} + 2^{-(r-1)n}]. \tag{4.9}$$

To this end, we need the following estimates:

$$\begin{aligned} &\text{(i)} \quad \|Q(y)-\bar{Q}(\bar{y})\|_{L^\infty(J)} \lesssim \|y-\bar{y}\|_{L^\infty(J)} + 2^{-rn},\\ &\text{(ii)} \quad \|Q'(y)-\bar{Q}'(\bar{y})\|_{L^\infty(J)} \lesssim \|y-\bar{y}\|_{L^\infty(J)} + 2^{-(r-1)n},\\ &\text{(iii)} \quad \|T-\bar{T}\|_{L^\infty(J)} \lesssim \|y-\bar{y}\|_{L^\infty(J)} + 2^{-rn}. \end{aligned} \tag{4.10}$$

Proof of (4.10) (i). Let us denote $Q_e := s^{-1}tg_n'(c\,\cdot)$. Then

$$\|Q(y)-\bar{Q}(\bar{y})\|_{L^\infty(J)} \le \|Q(y)-Q_e(\bar{y})\|_{L^\infty(J)} + \|Q_e(\bar{y})-\bar{Q}(\bar{y})\|_{L^\infty(J)}.$$

It follows from (3.4) that

$$\|Q(y)-Q_e(\bar{y})\|_{L^\infty(J)} \le \frac{2\bar{m}}{m}\|y-\bar{y}\|_{L^\infty(J)}$$

and since $\bar{Q}$ coincides with $s^{-1}tg_{n-1}'(c\,\cdot)$ on $\bar{y}(J)$, we infer from (3.3) that

$$\|Q_e(\bar{y})-\bar{Q}(\bar{y})\|_{L^\infty(J)} \lesssim 2^{-nr}.$$

Proof of (4.10) (ii). The same argument can be applied here since (3.3) implies in particular that $\left\|g_n^{(3)}\right\|_{L^\infty(\Omega)}$ is bounded independantly of n as long as $r \ge 2$.

Proof of (4.10) (iii). By (4.10) (i), we have

$$\begin{aligned} \|T-\bar{T}\|_{L^\infty(J)} &\le \|y-\bar{y}\|_{L^\infty(J)} + \|Q(y)-\bar{Q}(\bar{y})\|_{L^\infty(J)}\\ &\lesssim 2^{-nr} + \|y-\bar{y}\|_{L^\infty(J)}. \end{aligned}$$

Proof of (4.8). Assume first that $T(J)\cap\bar{T}(J)=\emptyset$ and without loss of generality, that $a := \sup(T(J)) < \inf(\bar{T}(J))$. We extend R by setting $R_e(x) = R(a) + (x-a)R'(a)$ for $x \ge a$. Then

$$\|R'(T)-\bar{R}'(\bar{T})\|_{L^\infty(J)} \le \|R'(T)-R_e'(\bar{T})\|_{L^\infty(J)} + \|R_e'(\bar{T})-\bar{R}'(\bar{T})\|_{L^\infty(J)}.$$

Since $R_e'(\bar{T})$ is a constant over J, we have

$$\begin{aligned} \|R'(T)-R_e'(\bar{T})\|_{L^\infty(J)} &\le \|R''\|_{L^\infty(T(J))}|T(J)|\\ &\lesssim \|R'\|_{L^\infty(T(J))}\\ &\lesssim 1\\ &\lesssim |J|^{-1}\|T-\bar{T}\|_{L^\infty(J)}. \end{aligned} \tag{4.11}$$

Here the second inequality is $(\mathbf{P_2})$, the third inequality is $(\mathbf{A_2})$, and for the latter inequality, we note that since $T(J)$ and $\bar{T}(J)$ are disjoint, then using (4.3),

$$|J| \sim \min(|T(J)|, |\bar{T}(J)|) \le \|T-\bar{T}\|_{L^\infty(J)}.$$

On the other hand, $R_e-\bar{R}$ is a polynomial over $\bar{T}(J)$ and hence we can apply again $(\mathbf{P_2})$ and (4.3) to obtain

$$\begin{aligned} \|R_e'-\bar{R}'\|_{L^\infty(\bar{T}(J))} &\lesssim |J|^{-1}\|R_e-\bar{R}\|_{L^\infty(\bar{T}(J))}\\ &\lesssim |J|^{-1}[\|R_e(\bar{T})-R(T)\|_{L^\infty(J)} + \|R(T)-\bar{R}(\bar{T})\|_{L^\infty(J)}]\\ &\lesssim |J|^{-1}[\|T-\bar{T}\|_{L^\infty(J)} + \|y-\bar{y}\|_{L^\infty(J)}], \end{aligned}$$

where we used $(\mathbf{A_2})$ and (3.7) for the latter estimate. Together with (4.11) and (4.10) (iii), this proves (4.8) in the case where $T(J)$ and $\bar{T}(J)$ are disjoint.

Let $T(J) \cap \bar{T}(J) \neq \emptyset$ and set $K := T(J) \cup \bar{T}(J)$. By (4.3), K is an interval of length $\mathcal{O}(|J|)$. Applying $(\mathbf{P_1})$ and $(\mathbf{P_2})$, we obtain

$$\|R'\|_{L^\infty(K)} \lesssim \|R'\|_{L^\infty(T(J))} \lesssim 1$$

and

$$\|R''\|_{L^\infty(K)} \lesssim |J|^{-1}.$$

We then have

$$\|R'(T) - R'(\bar{T})\|_{L^\infty(J)} \lesssim |J|^{-1}\|T - \bar{T}\|_{L^\infty(J)}$$

and also

$$\begin{aligned}
\|R' - \bar{R}'\|_{L^\infty(\bar{T}(J))} &\lesssim |J|^{-1}\|R - \bar{R}\|_{L^\infty(\bar{T}(J))} \\
&\lesssim |J|^{-1}[\|R(T) - R(\bar{T})\|_{L^\infty(J)} + \|R(\bar{T}) - \bar{R}(\bar{T})\|_{L^\infty(J)}] \\
&\lesssim |J|^{-1}[\|R'\|_{L^\infty(K)}\|T - \bar{T}\|_{L^\infty(J)} + \|y - \bar{y}\|_{L^\infty(J)}] \\
&\lesssim |J|^{-1}[\|T - \bar{T}\|_{L^\infty(J)} + \|y - \bar{y}\|_{L^\infty(J)}].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|R'(T) - \bar{R}'(\bar{T})\|_{L^\infty(J)} &\leq \|R'(T) - R'(\bar{T})\|_{L^\infty(J)} + \|R' - \bar{R}'\|_{L^\infty(\bar{T}(J))} \\
&\lesssim |J|^{-1}[\|T - \bar{T}\|_{L^\infty(J)} + \|y - \bar{y}\|_{L^\infty(J)}].
\end{aligned}$$

In view of (4.10) (iii), this completes the proof of (4.8).

Proof of (4.9). Observe that $(\mathbf{A_2})$ guarantees the boundedness of R' on $T(J)$ and since R is also obviously bounded on $T(J)$, we can apply $(\mathbf{P_2})$ to obtain

$$\|R'\|_{L^\infty(T(J))} \lesssim |J|^{-1}.$$

Then using the definition of Θ, we have

$$\begin{aligned}
\|\Theta - \bar{\Theta}\|_{L^\infty(J)} &= \|R'(T)(1 - Q'(y)) - \bar{R}'(\bar{T})(1 - \bar{Q}'(\bar{y}))\|_{L^\infty(J)} \\
&\leq \|R'(T) - \bar{R}'(\bar{T})\|_{L^\infty(J)} \\
&\quad + \|R'(T)\|_{L^\infty(J)}\|Q'(y) - \bar{Q}'(\bar{y})\|_{L^\infty(J)} \\
&\quad + \|\bar{Q}'(\bar{y})\|_{L^\infty(J)}\|R'(T) - \bar{R}'(\bar{T})\|_{L^\infty(J)} \\
&\lesssim |J|^{-1}[\|y - \bar{y}\|_{L^\infty(J)} + 2^{-(r-1)n}],
\end{aligned}$$

where the latter inequality follows from (4.8) and (4.10) (ii). This completes the proof of Lemma 4.1. □

4.2. *Proof of Lemma 3.2*

For simplicity, we denote $A' := A' \cdot \mathbb{1}_J$ and proceed to estimate $\|A'\|_{B^{\alpha-1}_{q,q}}$ following the approach of DeVore and Lucier [4]. Recall first the following inverse estimate [4, Lemma 4.3].

Lemma 4.2. *Let v be twice continuously differentiable on an open interval I and assume that v, v' and v'' each have one sign on I. If numbers p and q are given such* *that* $0 < p \le 1$ *and* $\frac{1}{p} - \frac{1}{q} > 1$, *then there exists a constant* C *such that whenever* $v \in L^q(I)$, *then* $v' \in L^p(I)$ *and*

$$\|v'\|_{L^p(I)} \le C\,|I|^{\frac{1}{p}-\frac{1}{q}-1}\,\|v\|_{L^q(I)}. \tag{4.12}$$

According to the definition of the Besov norm in (1.4)–(1.5), we have to estimate $\omega_k(A',t)_q := \sup_{|h|\le t} \left\|\Delta_h^k A'\right\|_{L^q(\mathbb{R})}$ for $t > 0$. Then because of symmetry, it suffices to bound $\left\|\Delta_h^k A'\right\|_{L^q}$ only for $0 < h \le t$. For a fixed $h > 0$, we introduce the following sets:

$$\Gamma := \{x \in \mathbb{R} : [x, x+kh] \subset J\}, \quad \Gamma' := \{x \in \mathbb{R} \setminus \Gamma : [x, x+kh] \cap J \neq \emptyset\}$$

and

$$\Gamma'' := \mathbb{R} \setminus (\Gamma \cup \Gamma') = \{x \in \mathbb{R} : [x, x+kh] \cap J = \emptyset\}.$$

If $x \in \Gamma''$, then $\Delta_h^k A'(x) = 0$ and hence

$$\int_{\Gamma''} \left|\Delta_h^k A'(x)\right|^q dx = 0. \tag{4.13}$$

If $x \in \Gamma'$, then using $\left|\Delta_h^k A'(x)\right| \le 2^k(|A'(x)| + \cdots + |A'(x+kh)|)$, we have

$$\int_{\Gamma'} \left|\Delta_h^k A'(x)\right|^q dx \le |\Gamma'|\,\left\|\Delta_h^k A'\right\|^q_{L^\infty(J)} \lesssim |\Gamma'|\,\|A'\|^q_{L^\infty(J)}.$$

Now, Lemma 4.2 and the obvious estimate $|\Gamma'| \le \min(h, |J|)$ yield

$$\int_{\Gamma'} \left|\Delta_h^k A'(x)\right|^q dx \lesssim \min(h,|J|)|J|^{-q}(\|A\|_{L^\infty(J)} + 2^{-(r-1)n})^q. \tag{4.14}$$

Finally, let $x \in \Gamma$ and $0 < h \le |J|/k$. Notice that $\Gamma = \emptyset$ if $h > |J|/k$. We shall employ the well known identity: $\Delta_h^k A'(x) = h^k A^{(k+1)}(\xi)$ for some $\xi \in [x, x+kh]$. From this and the monotonicity of $A^{(k+1)}$, we have

$$A^{(k+1)}(\xi) = h^k \max\{A^{(k+1)}(x), A^{(k+1)}(x+kh)\}.$$

Without loss of generality, we can assume that $A^{(k+1)}$ is decreasing. Then

$$\Delta_h^k A'(x) \le h^k A^{(k+1)}(x), \quad x \in \Gamma. \tag{4.15}$$

The following embedding is well known: If $1 < \beta_1 < \beta_2$, $q_j = 1/\beta_j$ and $f \in B^{\beta_2-1}_{q_2,q_2}$, then $f \in B^{\beta_1-1}_{q_1,q_1}$ and $\|f\|_{B^{\beta_1-1}_{q_1,q_1}} \lesssim \|f\|_{B^{\beta_2-1}_{q_2,q_2}}$. Therefore, we may assume that $k < \alpha < k+1$.

Set $q_0 := q = 1/\alpha$, $\varepsilon := \frac{1}{2}(\frac{\alpha}{k} - 1) > 0$ and define $q_1, q_2, \ldots, q_k$ recursively by the identity $\frac{1}{q_j} := \frac{1}{q_{j-1}} - (1+\varepsilon)$, $j = 1, \ldots, k$. Evidently, $\frac{1}{q_j} := \alpha - j(1+\varepsilon)$ and hence $\frac{1}{q_k} := \alpha - k(1+\varepsilon) = \frac{1}{2}(\alpha - k) > 0$. Therefore, $0 < q_0 < q_1 < \cdots < q_{k-1} < 1$ and

$q_k > 1$. Now, applying repeatedly Lemma 4.2, we obtain

$$\begin{aligned} \|A^{(k+1)}\|_{L^q(J)} &\lesssim |J|^{\varepsilon}\|A^{(k)}\|_{L^{q_1}(J)} \lesssim \cdots \lesssim |J|^{k\varepsilon}\|A'\|_{L^{q_k}(J)} \\ &\lesssim |J|^{k\varepsilon+1/q_k}\|A'\|_{L^\infty(J)} = c|J|^{1/q-\alpha}\|A'\|_{L^\infty(J)}. \end{aligned} \tag{4.16}$$

Using (4.15), (4.16) and Lemma 4.1, we get

$$\int_\Gamma \left|\Delta_h^k A'(x)\right|^q dx \lesssim h^{kq}|J|^{1-q-kq}(\|A\|_{L^\infty(J)} + 2^{-(r-1)n})^q. \tag{4.17}$$

Combining (4.13), (4.14) and (4.17), we arrive at

$$\begin{aligned} \omega_k(A',t)_q^q &= \sup_{0<h\le t}\int_{\mathbb{R}} \left|\Delta_h^k A'(x)\right|^q dx \\ &\lesssim [\min(t,|J|) + t^{kq}|J|^{1-kq}\mathbb{1}(t)]|J|^{-q}(\|A\|_{L^\infty(J)} + 2^{-(r-1)n})^q, \end{aligned}$$

where $\mathbb{1} := \mathbb{1}_{[0,|J|/k]}$. Therefore,

$$\begin{aligned} \|A'\|_{B_{q,q}^{\alpha-1}}^q &= \int_0^\infty t^{-(\alpha-1)q-1}\omega_k(A',t)_q^q dt \\ &\lesssim [|J|^{-q}\int_0^{|J|} t^{q-1}dt + |J|^{1-q}\int_{|J|}^\infty t^{q-2}dt \\ &\quad + |J|^{1-q-kq}\int_0^{|J|/k} t^{q+kq-2}dt](\|A\|_{L^\infty(J)} + 2^{-(r-1)n})^q \\ &\lesssim (\|A\|_{L^\infty(J)} + 2^{-(r-1)n})^q, \end{aligned}$$

where we used that $0 < q < 1$ and $kq + q - 2 = (k+1)/\alpha - 2 > -1$. The proof of Lemma 3.2 is complete. ◻

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