

1. (15 points) Let $A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Which of \mathbf{u} , \mathbf{v} , and \mathbf{w} is an eigenvector, and which is not? Explain, and give the corresponding eigenvalues, where appropriate.

$$A\vec{u} = \begin{bmatrix} -20 \\ 24 \end{bmatrix} = -4 \begin{bmatrix} 5 \\ -6 \end{bmatrix} = -4\vec{u} \quad \text{so } \vec{u} \text{ is an e-vec with eval } -4.$$

$$A\vec{v} = \begin{bmatrix} 13 \\ -3 \end{bmatrix} \neq \text{multiple of } \vec{v} \quad \text{so } \vec{v} \text{ is not an e-vec.}$$

$$A\vec{w} = \begin{bmatrix} 14 \\ 14 \end{bmatrix} = 7 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 7\vec{w} \quad \text{so } \vec{w} \text{ is an e-vec with eval } 7$$

2. (20 points) A population consists of individuals in four stages of development: newborns (N_t), juveniles (J_t), reproductive adults (R_t), and post-reproductive adults ("grandmothers" G_t). Newborns have a mortality rate of 70%; those that survive become juveniles. Juveniles have a total survival rate of 60% in two categories: 20% remain in the juvenile phase and 40% advance to the reproductive adult phase. Also juveniles rarely reproduce: on average each contributes 1 newborn to the next generation. Reproductive adults have a survival rate of 80% as reproductive adults, and 10% become post-reproductive adults. Meanwhile each contributes 5 newborns to the next generation. Post-reproductive adults produce no offspring, but have a survival rate of 70%.

- a. Set up the population transition matrix A to express the information above, so that $\mathbf{P}_{t+1} = A\mathbf{P}_t$.

$$A = \begin{bmatrix} 0 & 1 & 5 & 0 \\ 0.3 & 0.2 & 0 & 0 \\ 0 & 0.4 & 0.8 & 0 \\ 0 & 0 & 0.1 & 0.7 \end{bmatrix}$$

- b. The initial population vector is $\mathbf{P}_0 = \begin{bmatrix} 100 \\ 10 \\ 10 \\ 10 \end{bmatrix}$. Compute \mathbf{P}_1 .

$$\vec{P}_1 = A\vec{P}_0 = \begin{bmatrix} 1(10) + 5(10) \\ (0.3)(100) + (0.2)(10) \\ (0.4)(10) + (0.8)(10) \\ (0.1)(10) + (0.7)(10) \end{bmatrix} = \begin{bmatrix} 60 \\ 32 \\ 12 \\ 8 \end{bmatrix}$$

- c. At $t = 10$ we have $\mathbf{P}_{10} = \begin{bmatrix} 883 \\ 238 \\ 185 \\ 30 \end{bmatrix}$ (approximately). Determine the total population and the distribution vector \mathbf{D}_{10} . Carry three significant figure accuracy after the "leading" 0's (like 0.0273).

$$\text{Total}_{10} = 1336$$

$$\vec{D}_{10} = \begin{bmatrix} 0.661 \\ 0.178 \\ 0.138 \\ 0.0225 \end{bmatrix}$$

- d. The dominant eigenvalue is $\lambda = 1.3147$ with eigenvector $\mathbf{v} = \begin{bmatrix} 0.661 \\ 0.178 \\ 0.138 \\ 0.0225 \end{bmatrix}$.

Has the population reached its stable age/stage distribution at $t = 10$? How can you tell? Use λ to predict the total population at $t = 11$ and the value of N_{11} , that is, the number of newborns at $t = 11$; show your work.

Yes, because the \vec{D}_{10} agrees with the dominant eigenvalue's e-vec \vec{v} , which gives the SAD.

$$\begin{aligned} \text{Total}_{11} &\stackrel{\uparrow}{=} \lambda \cdot \text{Total}_{10} = (1.3147)(1336) \\ &\text{at SAD} \qquad \qquad \qquad = 1756 \\ N_{11} &\stackrel{\downarrow}{=} \lambda \cdot N_{10} = (1.3147)(883) = 1161 \end{aligned}$$