MATH 172 Fall, 2011 Exam #3 Name: M^2

There are 100 points. **For full credit you must show your work.** Recall that the geometric series $\sum_{n=0}^{\infty} ar^n$ has a sum $S_\infty = a/(1-r)$ under a certain condition on r, which you should verify, and fails to exist otherwise.

1. (12 points) Compute the equilibrium point (u^*, v^*) of the discrete system

$$\begin{aligned} u_n &= 2u_{n-1} - v_{n-1} - 1 \\ v_n &= u_{n-1} - 3v_{n-1} + 8 \end{aligned}$$

$$\begin{cases} u^* = 2u^* - v^* - 1 \\ v^* = u^* - 3v^* + 8 \end{cases}$$

$$\begin{cases} 1 = u^* - v^* \\ -8 = u^* - 4v^* \end{cases}$$

$$\begin{cases} 1 = u^* - v^* \\ 8 = -u^* + 4v^* \end{cases}$$

$9 = 3v^*$

$v^* = 3$

$u^* = 4$

$\boxed{(4,3)}$

2. (8 points) A reproductive female in the oldest stage of development produces 5 offspring on average each year. Her annual survival rate is 80%. What is her expected lifetime production of offspring?

$$5 + 5(0.8) + 5(0.8)^2 + \cdots = \sum_{n=0}^{\infty} 5(0.8)^n$$

$a = 5$

$r = 0.8$

$|0.8| < 1$

$$= \frac{5}{1-0.8} = \frac{5}{0.2} = \boxed{25}$$

3. (15 points) Find the sum if it does exist, or state that there is no sum, and why.

a. $\sum_{j=1}^{\infty}(2)(\frac{3}{7})^j = 2(\frac{3}{7}) + 2(\frac{3}{7})^2 + 2(\frac{3}{7})^3 + \cdots$

$$= 2(\tfrac{3}{7})(\tfrac{3}{7})^0 + (2)(\tfrac{3}{7})^1(\tfrac{3}{7})^1 + 2(\tfrac{3}{7})(\tfrac{3}{7})^2 + \cdots$$

$a = 2(\frac{3}{7})$

$r = \frac{3}{7}$ $|\frac{3}{7}| < 1$

$$= \frac{2 \cdot \frac{3}{7}}{1 - \frac{3}{7}} = \frac{\frac{6}{7}}{4/7} = \frac{6}{4} = \boxed{\frac{3}{2}}$$

b. $\sum_{n=0}^{\infty} \frac{2}{5}(-\frac{4}{3})^n$

$r = -\frac{4}{3}$ $|-\frac{4}{3}| > 1$

There is no sum

4. (15 points) Consider the following continuous model of a predator-prey system.

$$\frac{dV}{dt} = 0.7V(1-\frac{V}{350}) - 0.02VP = V\left(0.7(1-\frac{V}{350}) - 0.02P\right)$$
$$\frac{dP}{dt} = -0.9P + 0.003VP = P(-0.9 + 0.003V)$$

a. What kind of growth does the victim population exhibit if there are no predators (i.e., $P = 0$)?

$$\frac{dV}{dt} = 0.7V\left(1-\frac{V}{350}\right)$$

logistic growth
$r = 0.7$
$K = 350$

b. Why is $(V^*, P^*) = (350, 0)$ an equilibrium (mathematically), and how do you interpret this biologically?

Because both $\frac{dV}{dt}$ and $\frac{dP}{dt} = 0$. This is the case with no predators and prey (victims) at their carrying capacity.

c. Compute the equilibrium (V^*, P^*) other than $(0, 0)$ and $(350, 0)$ for this system.

$$(-0.9 + 0.003V) = 0$$
$$V^* = 0.9/0.003 = 300$$

$$0.7\left(1 - \frac{300}{350}\right) - 0.02P = 0$$

$$P = \frac{0.1}{0.02} = 5 \qquad (300, 5)$$

5. (15 points) A not very infectious illness, but one that is occasionally fatal, is spreading through a susceptible population $S(t)$. Each week there is a mass action interaction with transmission coefficient 0.02 of the susceptible population $S(t)$ with the ill and infectious population $I(t)$ in which susceptible individuals become ill and infectious. At the same time 45% of the infectious population recovers; the recovered population is denoted $R(t)$. Just 0.5% of the infected population dies. Most of the recovered population, in fact 80%, is immune to reinfection, but the rest do become susceptible again. Write a system of **continuous** model equations for for this process (the unit of measurement is thousands of people per week).

$$\frac{dS}{dt} = -0.02\,SI + 0.2R$$

$$\frac{dI}{dt} = 0.02\,SI - 0.45I - 0.005I$$

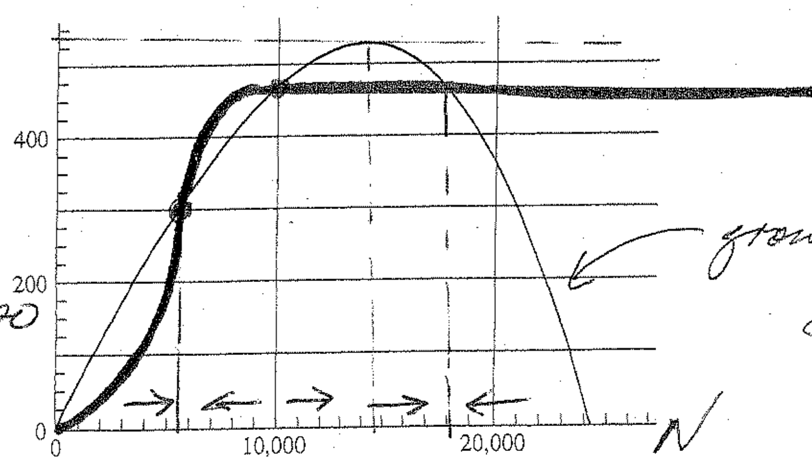
$$\frac{dR}{dt} = 0.45I - 0.2R$$

6. (15 points) A victim population has a growth rate curve (light line) as shown. Superimposed on this graph is a heavy curve indicating the loss rate due to moderate predation by a predator that exhibits a type III functional response.

a. Find the numerical values of the the non-zero equilibrium values for N^* on the graph, determine if each is stable or unstable, and indicate by arrows how N will change if it falls just slightly off each equilibrium value.

b What is the maximum sustainable predation level, and what is N for this level?

Max predation level = 530 at $N = 14{,}500$



(type III predation rate)

growth rate $\frac{dN}{dt}$

N

5,500

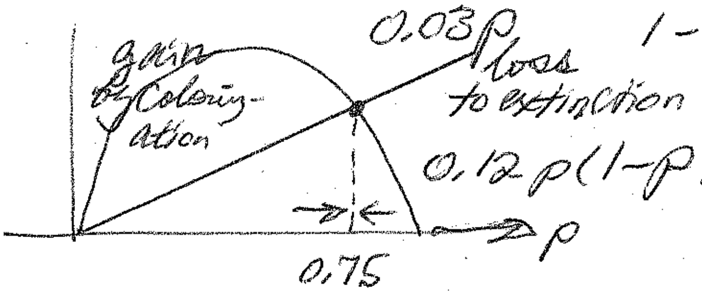
$N^* = 5{,}500$; $10{,}000$; $18{,}100$
stable ; unstable ; stable

7. (12 points) A metapopulation has a patch to patch colonization rate of 12% and a local extinction rate of 3%; that is $\frac{dp}{dt} = 0.12p(1-p) - 0.03p$. Find the equilibrium value other than $p = 0$ for the percent of occupied patches. Use a grpahical analysis to determine if this equilibrium is stable or not.

$$0 = \frac{dp}{dt} = 0.12p(1-p) - 0.03p$$

$$p = 0 \quad \text{or} \quad (0.12(1-p) - 0.03) = 0$$

$$1 - p = \frac{0.03}{0.12} = 0.25$$



$$p^* = 0.75$$

stable

8. (8 points) We have a successional model of species A, B, C, D. Time is measured in decades.

$$\begin{bmatrix} 0.2 & 0.1 & 0.1 & 0.1 \\ 0.7 & 0.4 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.4 & 0.3 \\ 0.0 & 0.2 & 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$$

a. If the current situation is 50% A and 50% B, what is the distribution in one decade?

$$\begin{bmatrix} .15 \\ .55 \\ .2 \\ .1 \end{bmatrix}$$

15% A   20% C

55% B   10% D

b. The dominant eigenvalue for this matrix is 1, which has eigenvector $\mathbf{v} = \begin{bmatrix} 0.111 \\ 0.282 \\ 0.309 \\ 0.298 \end{bmatrix}$. What information does this give you? Is there a climax species, that is, one that dominates the habitat in the long term?

In the long term there is 11.1% A, 28.2% B, 30.9% C, 29.8% D. No single species dominates this habitat, so no climax species.