MATH 172 **Fall, 2007** **Exam #2** **Name:** M

There are 100 points. **For full credit you must show your work.** All answers should be **correctly rounded to 3 decimal places**, or fewer if that is what is provided in the data. Recall that an affine continuous dynamic model $\frac{dQ}{dt} = aQ + b$ has an explicit solution $Q(t) = Ce^{at} + Q^*$, where Q^* is the equilibrium value, and C can be determined from the initial condition. Very basic model equations can be solved from memory; otherwise use separation of variables and integration, graphical analysis, or Euler's method, as appropriate.

1. (15 points) Somalia currently has a population of 9.119 million people. The per capita growth rate is 29.2 people per 1000 people per year. Write a continuous dynamic model equation for the net growth rate of Somalia. Give the explicit solution and use your result to predict the population a decade from now.

$$\boxed{\frac{dP}{dt} = \frac{29.2}{1000}P = 0.0292P}$$ model

From memory, or separation of variables, etc.

Sol'n: $P(t) = P(0)\cdot e^{0.0292t} = 9.119e^{0.0292t}$

At $t=10$: $P(10) = 12.211$ million people

2. (10 points) If $\frac{dW}{dt} = 0.3t^2$ and we have a data point $W(10) = 60$, determine an explicit formula for $W(t)$.

Separate variables: $dW = 0.3t^2\,dt$

Integrate $W = \int dW = \int 0.3t^2\,dt$

$= 0.3\left(\frac{t^3}{3}\right) + A$

$W(t) = 0.1t^3 + A$

$W(10) = 60 = (0.1)(1000) + A = 100 + A$

$A = -40$ $\quad W(t) = 0.1t^3 - 40$

3. (25 points) A grasshopper population $G = G(t)$, measured in millions, now finds that the quality of the habitat is decreasing as measured by the per capita growth rate $s(t)$; in fact $\frac{dG}{dt} = s(t)G$, with $s(t) = 0.08 - 0.005t$ in units of yr^{-1}. At time $t = 0$ there were 20 million individuals.

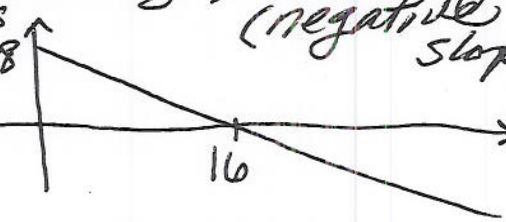
a. How does the information provided tell us that the quality of the habitat is decreasing?

$s(t)$ is a linear decreasing function of t. (negative slope)

$s(t) = 0$

$0.08 = 0.005t$

$t = 16$ years



b. Solve the dynamic model equation and use your solution to find the number of grasshoppers at time $t = 10$ years.

Separate variables: $\frac{dG}{G} = s(t)\,dt$

$$\int \frac{dG}{G} = \int (0.08 - 0.005t)\,dt = 0.08t - 0.005\left(\frac{t^2}{2}\right) + A$$

$\ln G$

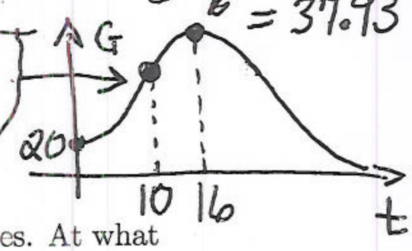
Integrate, then take exp. on both sides. Use $C = e^A$.

$$G(t) = Ce^{0.08t - 0.005\left(\frac{t^2}{2}\right)}$$

$$G(0) = 20 = Ce^0 = C$$

$$\boxed{G(t) = 20e^{0.08t - 0.0025t^2}}$$

$$\boxed{G(10) = 34.665}$$



sign hinges on sign of $s(t)$

c. Explain how you can tell that $G(t)$ increases and then decreases. At what time does the population peak?

$\frac{dG}{dt} = s(t)G$

positive

(1) $s(t) > 0$ for $0 \le t < 16$ so $\frac{dG}{dt} > 0$ and $G(t)$ is increasing.

(2) $s(t) < 0$ for $t > 16$ so $\frac{dG}{dt} < 0$ and $G(t)$ is decr.

(3) $G(t)$ peaks at $t = 16$

4. (25 points) An algal population $A(t)$, measured in tons, grows in some lakes and rivers that accumulate nutrients from agricultural runoff. It is governed by the continuous dynamic model $\frac{dA}{dt} = 0.0005A(150 - A) = 0.075A(1 - \frac{A}{150})$ tons/year. The initial population is $A(0) = 30$ tons.

a. What is the significance of each of the numbers 0.075 and 150 in this model?

logistic model

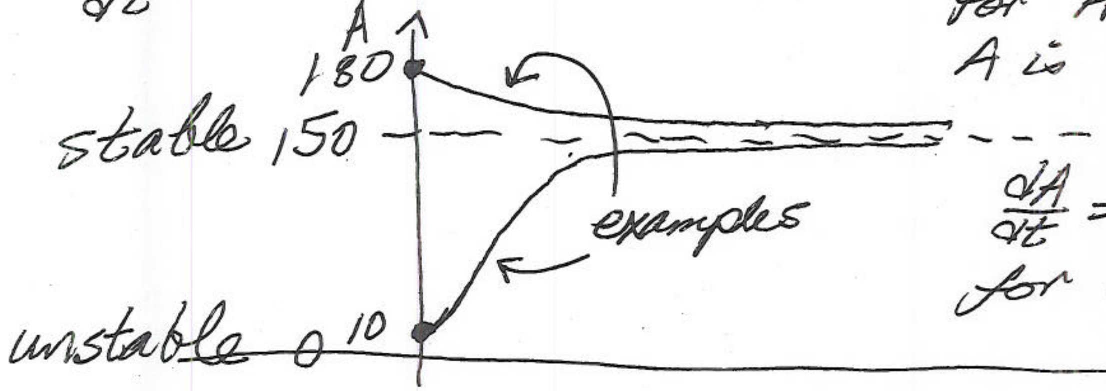
r = per capita growth rate = 0.075

K = carrying capacity = 150

b. Find all equilibrium values for this population, and determine if each one is stable or unstable.

$\frac{dA}{dt} = 0$ at $A^* = 0$, $A^* = 150$

$\frac{dA}{dt} = (+)(+)(-) = (-)$ for $A > 150$, so A is decreasing



$\frac{dA}{dt} = (+)(+)(+) = (+)$ for $0 < A < 150$ so A is increasing

c. It is easy to separate the variables of this model equation, but hard to do the integration, so we apply stepwise estimation using Euler's method to estimate the population after 10 years. Remember that $\Delta A \approx \frac{dA}{dt}(\Delta t)$. Use 2 steps (so $\Delta t =$ <u>5</u>) to approximate $A(10)$, and show your work!

n	t	Δt	A	$\frac{dA}{dt}$	$\Delta A \approx \frac{dA}{dt}\Delta t$
0	0	5	30	1.8	9
1	5	5	39	2.1645	10.8225
2	10		49.8225		

rounds to 49.822

or 49.823

d. We are still using $\frac{dA}{dt} = 0.0005A(150 - A) = 0.075A(1 - \frac{A}{150})$ tons/year with $A(0) = 30$ tons. Approximate $A(10)$ again, but use 40 steps (so $\Delta t =$ 0.25). For partial credit, indicate what you are putting into your calculator.

$\frac{10}{40} = \frac{1}{4}$

$A(10) \approx 51.803$

nMin = 0

u(nMin) = 30

Ask for step n = 40.

$u(n) = u(n-1) + (0.0005)u(n-1)(150 - u(n-1))(0.25)$

5. (25 points) In a particular wooded habitat, a population of birds $B = B(t)$ **declines** at a **per capita rate** of 6% yr $^{-1}$, but is reinforced by the in-migration of birds from destroyed nearby habitats at 300 birds/yr. Write a continuous dynamic model equation for this situation, and solve it, assuming that the initial bird population is 15,000. What exactly happens to the bird population in the long term, and how do you know?

$$\boxed{\frac{dB}{dt} = -0.06B + 300}$$ affine model

$0 = -0.06B + 300$

$B^* = 5000$ equilibrium

$B(t) = Ce^{-0.06t} + 5000$

$B(0) = 15000 = Ce^{0} + 5000$

$C = 10,000$

$$\boxed{B(t) = 10,000e^{-0.06t} + 5,000}$$

$e^{-0.06t}$ is a decay exp., so $\to 0$ as $t \to \infty$, Thus $B(t) \to 5000 = B^*$ as $t \to \infty$.