MATH 544 (Section 501)
Prof. Meade

University of South Carolina
Spring 2010

Exam 1
19 February 2010

Name: Key
SS # (last 4 digits): ____________

Instructions:

1. There are a total of 6 problems (not counting the Extra Credit problem) on 6 pages. Check that your copy of the exam has all of the problems.
2. No electronic or other inanimate objects can be used during this exam. All questions have been designed with this in mind and should not involve unreasonable manual calculations.
3. Be sure you answer the questions that are asked.
4. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
5. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
6. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	18	
2	16	
3	20	
4	20	
5	16	
6	10	
Extra Credit	10	
Total	100	

Good Luck!

1. (18 points) Identify each statement as either True or False. You do *not* have to justify your answer to receive credit for a correct answer, but some explanation might make you eligible for some partial credit for an incorrect answer.

   (a) __F__ Any system of n linear equations in n variables has at most n solutions. could have an ∞ # of sol'ns.

   (b) __T__ If a system of linear equations has two different solutions, it must have an infinite number of solutions.

   (c) __F__ If a system of linear has no free variables, then it has a single solution. it has ~~at~~ most one solution.

   (d) __T__ If an augmented matrix $[A\ \mathbf{b}]$ is transformed into $[C\ \mathbf{d}]$ by elementary row operations, then the equations $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ have the same solution sets.

   (e) __F__ The equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if there are no free variables. The homogeneous equation always has the trivial sol'n.

   (f) __T__ If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbf{R}^m$, then A has m pivot columns. (pivot in every row) & there are m rows

   (g) __T__ If an $n \times n$ matrix A has n pivot positions, then the reduced echelon form of A is the $n \times n$ identity matrix.

   (h) __T__ If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are not in $\mathbf{R}^2$. 3 vectors in $\mathbb{R}^2$ must be lin. dependent.

   (i) __T__ If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{R}^m$, then $-\mathbf{u}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$.
   $-\vec{u} = -1\vec{u} + 0\vec{v}$ is a lin. comb. of $\vec{u}$ & $\vec{v}$.

2. (16 points) Determine whether each of the following formulas is True or False for all scalars a and b and all $n \times n$ matrices A, B, and C.

   (a) __T__ $(A + B)C = AC + BC$

   (b) __F__ $(A + B)(A - B) = A^2 - B^2$
   $(A+B)(A-B) = AA - AB + BA + BB$
   $= A^2 - AB + BA + B^2$

   (c) __T__ If $B = A - A^T$, then $B^T = -B$
   $B^T = (A - A^T)^T = A^T - A^{TT} = A^T - A = -(A - A^T)$
   $= -B.$

   (d) __F__ $(AB)^T = A^T B^T$ $(AB)^T = B^T A^T.$

   (e) __T__ $(AB)^{-1} = B^{-1}A^{-1}$

   (f) __T__ $(aA + bB)^T = aA^T + bB^T$

   (g) __F__ $(aA + bB)^{-1} = aA^{-1} + bB^{-1}$

   (h) __T__ If $AB = BA$ and if A is invertible, then $A^{-1}B = BA^{-1}$.
   $AB = BA$ so $A^{-1}AB = A^{-1}BA$
   $B = A^{-1}BA$ and then $BA^{-1} = A^{-1}BAA^{-1} = A^{-1}B.$

3. (20 points) Consider the system of equations $\begin{array}{rcrcrcr} 3x & + & 5y & - & 2z & = & 17 \\ x & + & y & & & = & 5 \\ & & 3y & - & 3z & = & p \end{array}$.

(a) Write the augmented matrix, then perform appropriate row operations to obtain the row echelon form.

$$\left[\begin{array}{ccc|c} 3 & 5 & -2 & 17 \\ 1 & 1 & 0 & 5 \\ 0 & 3 & -3 & p \end{array}\right] \xrightarrow{①-3②\to②} \begin{bmatrix} 3 & 5 & -2 & 17 \\ 0 & 2 & -2 & 2 \\ 0 & 3 & -3 & p \end{bmatrix} \xrightarrow{\frac{1}{2}②} \begin{bmatrix} 3 & 5 & -2 & 17 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -3 & p \end{bmatrix}$$

$$\xrightarrow{③-3②\to③} \begin{bmatrix} 3 & 5 & -2 & 17 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & p-3 \end{bmatrix} \xrightarrow{①-5②\to①} \begin{bmatrix} 3 & 0 & 3 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & p-3 \end{bmatrix} \xrightarrow{\frac{1}{4}①} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & p-3 \end{bmatrix}$$

you can answer (b) now, but the matrix is not yet in reduced echelon form

(b) For what (if any) value(s) of p are there

NOTE: For at least one of these categories the answer is "No values of p".

i. no solutions?

$p \neq 3$ (so $p-3 \neq 0$).

ii. a unique solution?

never.

iii. exactly two solutions

never (you knew this before you did any work in (a))

iv. infinitely-many solutions?

$p=3$.

(c) For any value of p for which there are solutions, what is the solution set?

When $p=3$ the reduced echelon form is $\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The solutions are: $x_1 = 4 - x_3$
$x_2 = 1 + x_3$
$x_3 = x_3$

or $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4-x_3 \\ 1+x_3 \\ x_3 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

4. (20 points) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 \end{bmatrix}$.

NOTE: All entries of the inverse are integers.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{3 - 1 \to 3} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{3 \leftrightarrow 4} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[-4]{3 - 2\,2 \to 3} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & +1 & +1 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[1 - 4 \to 1]{3 - 3\,4 \to 3} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & -2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[2 - 3 \to 2]{1 - 3 \to 1} \begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 2 & -2 & -1 \\ 0 & 1 & 0 & 0 & 3 & 3 & -3 & -1 \\ 0 & 0 & 1 & 0 & -3 & -2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

so $A^{-1} = \begin{bmatrix} 3 & 2 & -2 & -1 \\ 3 & 3 & -3 & -1 \\ -3 & -2 & 3 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$

To check:

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 & -1 \\ 3 & 3 & -3 & -1 \\ -3 & -2 & 3 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underline{I}.$$

5. (16 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation such that $T(x_1, x_2, x_3) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$.

$$T(\vec{x}) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \vec{x}.$$

(a) Show that $\begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}$ is in the image set of T.

To show $\vec{b} = \begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}$ is in the image set of T, we show there is a solution to $\begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix} \xrightarrow[(3)-3(1)\to(3)]{(2)+(1)\to(2)} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix} \xrightarrow{(3)-4(2)\to(3)} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Because the system is consistent, there is a solution.

The solution is: $x_2 = 3$

$x_1 = -1 + 2x_2 = -1 + 6 = 5.$

(b) Explain why this T is not onto $\mathbf{R}^3$.

T cannot be onto $\mathbb{R}^3$ because there is not a pivot in each of the 3 rows. (In fact, we could have known this before answering (a).)

6. (10 points) Let A be a 3×3 matrix with the property that the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbf{R}^3$ onto $\mathbf{R}^3$. Explain why the transformation must be one-to-one.

When the linear transformation is onto $\mathbb{R}^3$, there must be a pivot in each of the three rows of A. But, this means there must be a pivot in each of the three columns of A, and so this mapping is also one-to-one.

Extra Credit (10 points) Write the *reduced* echelon form of a 3×3 matrix A such that the first two columns of A are pivot columns and $A \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The reduced echelon form of a 3x3 matrix with pivots in the first 2 columns is

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{bmatrix}.$$

Because there is a nontrivial solution to the homogeneous equation, there must be at least one free variable. So, $c = 0$.

Also, $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ must be a solution to $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

so $\begin{bmatrix} 3+a \\ -2+b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We now see that $a = -3$ and $b = 2$, and the reduced echelon form is $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.