HW Solns for §3.3.1

#1. $u_{tt} = 9(u_{xx}+u_{yy})$ $\quad 0<x<3,\ 0<y<6$

$u(x,0,t) = u(x,3,t) = 0 \qquad 0<x<3,\ t>0$

$u(0,y,t) = u(3,y,t) = 0 \qquad 0<y<3,\ t>0$

$$\left.\begin{array}{l} u(x,y,0) = \sin\left(\frac{\pi x}{3}\right)y(6-y) \\ u_t(x,y,0) = 0 \end{array}\right\} \quad 0<x<3,\ 0<y<6$$

Let $u(x,y,t) = X(x)Y(y)T(t)$: $u_{tt} = 9(u_{xx}+u_{yy})$ becomes $\frac{XYT''}{9XYT} = \frac{9(X''YT+XY''T)}{9XYT}$

so $\frac{T''}{9T} = \frac{X''}{X} + \frac{Y''}{Y}$. Rewrite as $\frac{T''}{9T} - \frac{Y''}{Y} = \frac{X''}{X} = -\lambda$.

Thus: $X''+\lambda X = 0,\ X(0) = X(3) = 0 \Rightarrow X_n(x) = \sin\left(\frac{n\pi x}{3}\right)$

with $\lambda_n = -\left(\frac{n\pi}{3}\right)^2$.

Next, $\frac{T''}{9T} + \lambda = \frac{Y''}{Y} = -\mu$ so $Y''+\mu Y = 0$, with $Y(0) = Y(3) = 0$.

Nontrivial solutions are $Y_m(y) = \sin\left(\frac{m\pi y}{3}\right)$ with $\mu_m = -\left(\frac{m\pi}{3}\right)^2$.

Now, $\frac{T''}{9T} = -\lambda-\mu$ so $T''+9(\lambda+\mu)T = T''+9\left(\left(\frac{n\pi}{3}\right)^2+\left(\frac{m\pi}{3}\right)^2\right)T = 0$

$T''+(n^2+m^2)\pi^2 T = 0.$

so $T(t) = a\cos\left(\sqrt{n^2+m^2}\,\pi t\right) + b\sin\left(\sqrt{n^2+m^2}\,\pi t\right)$.

but $u_t(x,y,0) = 0 \Rightarrow T'(0) = 0$: $T'(t) = -a\sqrt{n^2+m^2}\,\pi\sin\left(\sqrt{n^2+m^2}\,\pi t\right)$

$+ b\sqrt{n^2+m^2}\,\pi\cos\left(\sqrt{n^2+m^2}\,\pi t\right)$

$T'(0) = b\sqrt{n^2+m^2}\,\pi = 0 \Rightarrow b = 0.$

$\therefore T_{mn}(t) = \cos\left(\sqrt{n^2+m^2}\,\pi t\right).$

The general solution is $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}\sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{m\pi y}{3}\right)\cos\left(\sqrt{n^2+m^2}\,\pi t\right)$

Now $u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}\sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{m\pi y}{3}\right) = \sin\left(\frac{\pi x}{3}\right)y(6-y)$

By inspection, $b_{mn} = 0$ for all $n = 2,3,\ldots$, and $m = 1,2,3,\ldots$

And $b_{m1} = \frac{4}{3\cdot 3}\int_0^3\int_0^3 \sin\left(\frac{\pi x}{3}\right)y(6-y)\sin\left(\frac{\pi x}{3}\right)\sin\left(\frac{m\pi y}{3}\right)dy = \frac{2}{3}\int_0^3 y(6-y)\sin\left(\frac{m\pi y}{3}\right)d$

$$= \begin{cases} \frac{-18}{m\pi} & m \text{ ev} \\ \frac{+18}{m\pi} + \frac{72}{m^3\pi^3} & m\text{o} \end{cases}$$

Putting everything together, we have the final solution as:

$$\boxed{\begin{aligned} u(x,y,t) &= \sum_{m=1}^{\infty} b_{m1} \sin\left(\frac{\pi x}{3}\right) \sin\left(\frac{m\pi y}{3}\right) \cos\left(\sqrt{1+m^2}\,\pi t\right) \\ \text{where } b_{m1} &= \begin{cases} \frac{-18}{m\pi} & \text{if } m \text{ is even} \\ \frac{+18}{m\pi} + \frac{72}{m^3\pi^3} & \text{if } m \text{ is odd.} \end{cases} \end{aligned}}$$

#3. We are considering the BVP:

$$u_{tt} = c^2(u_{xx}+u_{yy}) \qquad 0<x<L,\ 0<y<K \quad t>0$$

$$u(x,0,t) = u(x,K,t) = 0 \qquad 0<x<L \quad t>0$$

$$u(0,y,t) = u(L,y,t) = 0 \qquad 0<y<K \quad t>0$$

$$\left.\begin{aligned} u(x,y,0) &= 0 \\ u_t(x,y,0) &= \psi(x,y). \end{aligned}\right\} \ 0<x<L,\ 0<y<K$$

The separation of variables begins as in the case with $u(x,y,0)=\varphi(x,y)$ and $u_t(x,y,0)=0$, finding

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{with } \lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

and $$Y_m(y) = \sin\left(\frac{m\pi y}{k}\right) \quad \text{with } \mu_n = -\left(\frac{n\pi}{k}\right)^2$$

The equation for T is the same: $T'' + c^2\left(\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{k}\right)^2\right)^{1/2} T = 0$

but the condition that $u(x,y,0)=0$ requires $T(0)=0$ and so we find

$$T_{mn}(t) = \underbrace{\sin}_{\text{this changes from cos to sin.}}\left(c\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{k}\right)^2}\ t\right)$$

Then $$\boxed{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{k}\right) \sin\left(c\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{k}\right)^2}\ t\right)}$$

where $$u_t(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{k}\right)^2}\ b_{mn} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{k}\right) \cancelto{1}{\cos\left(c\sqrt{\cdots}\ 0\right)} = \psi(x,y)$$

i.e. $$b_{mn} = \frac{1}{c\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{k}\right)^2}}\ \frac{4}{L\cdot k} \int_0^L \int_0^K \psi(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{k}\right) dy\,dx$$