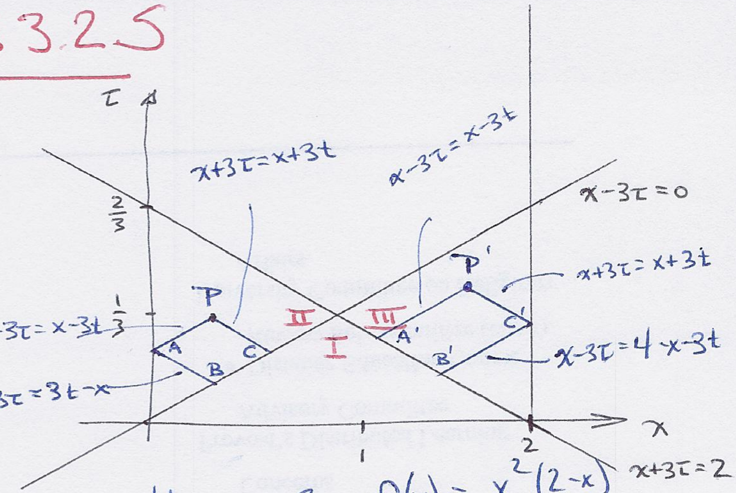
HW Sol'n for §3.2.5

#3. $u_{tt} = 9u_{xx} \quad 0<x<2, \; t>0$

$$\left.\begin{aligned} u(x,0) &= x^2(2-x) \\ u_t(x,0) &= 0 \end{aligned}\right\} \; 0<x<2$$

$$\left.\begin{aligned} u(0,t) &= 18t^2 \\ u(2,t) &= -36t^2 \end{aligned}\right\} \; t>0$$



In Region I we use d'Alembert's formula with $c=3$, $\varphi(x) = x^2(2-x)$ and $\psi(x)=0$:

$$u(x,t) = \frac{1}{2}\left(\varphi(x+3t)+\varphi(x-3t)\right) + \frac{1}{2\cdot 3}\int_{x-3t}^{x+3t} \psi(s)\,ds$$

$$= \frac{1}{2}\left((x+3t)^2(2-(x+3t)) + (x-3t)^2(2-(x-3t))\right) + \frac{1}{6}\cdot 0$$

$$= \frac{1}{2}\left((x^2+6xt+9t^2)(2-x-3t) + (x^2-6xt+9t^2)(2-x+3t)\right)$$

$$= \frac{1}{2}\Big(2x^2 - x^3 - \cancel{3x^2t} + \cancel{12xt} - \cancel{6x^2t} - 18xt^2 + 18t^2 - 9xt^2 - \cancel{27t^3}$$
$$+ 2x^2 - x^3 + \cancel{3x^2t} - \cancel{12xt} + \cancel{6x^2t} - 18xt^2 + 18t^2 - 9xt^2 + \cancel{27t^3}\Big)$$

$$= \boxed{2x^2 - x^3 - 27xt^2 + 18t^2}$$

For Region II, let P be a point (x,y). Then $u(P) = u(A)+u(C)-u(B)$.
We just have to determine the coordinates of A, B, and C.

A: $\begin{aligned} x-3\tau &= x-3t \\ x &= 0 \\ \hline +3\tau &= -x+3t \end{aligned}$ $\quad \begin{cases} \tau = (3t-x)/3 \\ x = 0 \end{cases}$

B: $\begin{aligned} x+3\tau &= 3t-x \\ x-3\tau &= 0 \\ \hline 2x &= 3t-x \\ 6\tau &= 3t-x \end{aligned}$ $\Rightarrow \begin{cases} x = (3t-x)/2 \\ \tau = (3t-x)/6. \end{cases}$

C: $\begin{aligned} x+3\tau &= x+3t \\ x-3\tau &= 0 \\ \hline 2x &= x+3t \\ 6\tau &= x+3t \end{aligned}$ $\Rightarrow \begin{cases} x = (x+3t)/2 \\ \tau = (x+3t)/6 \end{cases}$

Now, $u(x,t) = u(A)+u(C)-u(B) = u\left(0, \frac{(3t-x)}{3}\right) + u\left(\frac{x+3t}{2}, \frac{x+3t}{6}\right) - u\left(\frac{3t-x}{2}, \frac{3t-x}{6}\right)$

$$= 18\left(\frac{3t-x}{3}\right)^2 + \left(2\left(\frac{x+3t}{2}\right)^2 - \left(\frac{x+3t}{2}\right)^3 - 27\left(\frac{x+3t}{2}\right)\left(\frac{x+3t}{6}\right)^2 + 18\left(\frac{x+3t}{6}\right)^2\right)$$
$$- \left(2\left(\frac{3t-x}{2}\right)^2 - \left(\frac{3t-x}{2}\right)^3 - 27\left(\frac{3t-x}{2}\right)\left(\frac{3t-x}{6}\right)^2 + 18\left(\frac{3t-x}{6}\right)^2\right)$$

Maple helped here!

$$= \boxed{2x^2 - x^3 - 27xt^2 + 18t^2}$$

For Region III, let P' be a point (x,t). Then $u(P') = u(A') + u(C') - u(B')$.
We just have to find the coordinates of A', B', and C'.

A':
$$\begin{aligned} \chi - 3\tau &= x - 3t \\ \chi + 3\tau &= 2 \\ \hline 2\chi &= 2 + x - 3t \\ 6\tau &= 2 - x + 3t \end{aligned}$$

$$\begin{cases} \chi = \dfrac{2+x-3t}{2} \\ \tau = \dfrac{2-x+3t}{6} \end{cases}$$

C':
$$\begin{aligned} \chi + 3\tau &= x + 3t \\ \chi &= 2 \\ \hline 3\tau &= x + 3t - 2 \end{aligned}$$

$$\begin{cases} \chi = 2 \\ \tau = \dfrac{x+3t-2}{3} \end{cases}$$

B':
$$\begin{aligned} \chi + 3\tau &= 2 \\ \chi - 3\tau &= 4 - x - 3t \\ \hline 2\chi &= 6 - x - 3t \\ 6\tau &= -2 + x + 3t \end{aligned}$$

$$\begin{cases} \chi = \dfrac{6-x-3t}{2} \\ \tau = \dfrac{-2+x+3t}{6} \end{cases}$$

Now, $u(x,t) = u(A') + u(C') - u(B') = \underbrace{u\left(\dfrac{2+x-3t}{2}, \dfrac{2-x+3t}{6}\right)}_{\text{from region I}} + \underbrace{u\left(2, \dfrac{x+3t-2}{3}\right)}_{\text{from BC @ } x=2} - \underbrace{u\left(\dfrac{6-x-3t}{2}, \dfrac{-2+x+3t}{6}\right)}_{\text{from region I}}$

$= \ldots$ (Maple)

$$= \boxed{2x^2 - x^3 - 27xt^2 + 18t^2}$$

Note: It's a little anti-climatic to get the same answer all 3 times. This happens because the BC are scalar multiples. If, say, the BC at $x=2$ was changed to $u(2,t) = -12t$ then the solutions in I and II would be the same, but the one in Region III would become:

$$u(x,t) = 6x^2 - x^3 - 20x - 27xt^2 + 24xt + 54t^2 - 60t + 24.$$

Note that along the common boundary between I and III ($x + 3\tau = 2$) both regions' solution are $108t^3 - 72t^2 + 12t$ showing (at least) continuity!