HW Solution for §2.3

#2. $u_t = u_{xx} + u_{yy}$ $\quad 0<x<1,\ 0<y<1,\ t>0$

$u_x(0,y,t) = u_x(1,y,t) = 0 \quad 0<y<1,\ t>0$

$u_y(x,0,t) = u_y(x,1,t) = 0 \quad 0<x<1,\ t>0$

$u(x,y,0) = K \quad 0<x<1,\ 0<y<1$

This is the basic heat equation in two variables with $a=b=1$ and $k=1$. All 4 sides are insulated.

Separation of variables $u(x,y,t) = X(x)Y(y)T(t)$ will lead to

$X'' + \lambda X = 0$, $X'(0) = X'(1) = 0$ $\Downarrow$ $X_m(x) = \cos(m\pi x)$, $\lambda_m = (m\pi)^2$ $(m=0,1,2,\ldots)$

$Y'' + \mu Y = 0$, $Y'(0) = Y'(1) = 0$ $\Downarrow$ $Y_n(y) = \cos(n\pi y)$, $\mu_n = (n\pi)^2$ $(n=0,1,2,\ldots)$

$T' + (\lambda+\mu)T = 0$ $\Downarrow$ $T_{mn} = e^{-(\lambda_m+\mu_n)t} = e^{-(m^2+n^2)\pi^2 t}$

The general solution is composed from

$$u_{mn}(x,y,t) = X_m(x)Y_n(y)T_{mn}(t).$$

To have all of the coefficients work out with the usual Fourier coefficient formulas the solution needs to be written as:

$$u(x,y,t) = \underbrace{\frac{C_{00}}{4}}_{m=n=0} + \sum_{n=1}^{\infty} \underbrace{\frac{C_{0n}}{2}}_{m=0} \cos(n\pi y)e^{-n^2\pi^2 t} + \sum_{m=1}^{\infty} \underbrace{\frac{C_{m0}}{2}}_{n=0} \cos(m\pi x)e^{-m^2\pi^2 t} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\cos(m\pi x)\cos(n\pi y)\, e^{-(n^2+m^2)\pi^2 t}$$

To satisfy the initial condition: $u(x,y,0) = K$ means

$$K = \frac{C_{00}}{4} + \sum_{n=1}^{\infty}\frac{C_{0n}}{2}\cos(n\pi y) + \sum_{m=1}^{\infty}\frac{C_{m0}}{2}\cos(m\pi x) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\cos(m\pi x)\cos(n\pi y).$$

By inspection we can see that the LHS is a Fourier cosine series in both x & y that has only the constant term. So, it must be the case that

$$\frac{C_{00}}{4} = K \quad \text{and all other } C_{mn} = 0.$$

(see p. 2 for a more thorough explanation).

Thus, $u(x,y,t) = K$ is the final and complete solution.

To see how the formula

$$C_{mn} = \frac{2}{1}\int_0^1 \frac{2}{1}\int_0^1 K\cos(m\pi x)\,dx\,\cos(n\pi y)\,dy$$

gives $C_{00} = 4K$ and $C_{mn} = 0$ for all other values of m and n:

① if $m=n=0$ then $C_{00} = 4\int_0^1\int_0^1 K\,dx\,dy = 4K.$

② if $m=0, n\geq 1$ then $C_{0n} = 4\int_0^1\int_0^1 K\cos(n\pi y)\,\cancel{dy}\,dx\,dy = 4\int_0^1 K\cos(n\pi y)\,dy$

$$= \frac{4K}{n\pi}\sin(n\pi y)\Big|_0^1 = \frac{4K}{n\pi}\left(\sin(n\pi) - \sin(0)\right) = 0.$$

③ if $m\geq 1, n=0$ then $C_{m0} = 4\int_0^1\int_0^1 K\,\cancel{dx}\,dy\,\cos(m\pi x)\,dx = 4\int_0^1 K\cos(m\pi x)\,dx$

$$= \frac{4K}{m\pi}\sin(m\pi x)\Big|_0^1 = \frac{4K}{m\pi}\left(\sin(m\pi) - \sin(0)\right) = 0.$$

④ if $m\geq 1, n\geq 1$ then $C_{mn} = 4\int_0^1\int_0^1 K\cos(m\pi x)\cos(n\pi y)\,dx\,dy$

$$= 4K\int_0^1 \frac{\sin(m\pi x)}{m\pi}\Big|_0^1 \cos(n\pi y)\,dy$$

$$= 4K\,\frac{\sin(m\pi)-\sin(0)}{m\pi}\int_0^1 \cos(n\pi y)\,dy$$

$$= 0.$$

Since I have some extra space, let me fill it with a little bit of an explanation for why I wrote the solution with $\frac{C_{00}}{4}$, $\frac{C_{m0}}{2}$, $\frac{C_{0n}}{2}$, and C_{mn} as the coefficients:

$$\frac{C_{00}}{4} + \sum_{n=1}^{\infty}\frac{C_{0n}}{2}\cos(n\pi y) + \sum_{m=1}^{\infty}\frac{C_{m0}}{2}\cos(m\pi x) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\cos(m\pi x)\cos(n\pi y)$$

$$= \frac{1}{2}\left(\frac{C_{00}}{2} + \sum_{n=1}^{\infty} C_{0n}\cos(n\pi y)\right) + \sum_{m=1}^{\infty}\left(\frac{C_{m0}}{2} + \sum_{n=1}^{\infty} C_{mn}\cos(n\pi y)\right)\cos(m\pi x)$$

$$= \frac{S_0(y)}{2} + \sum_{m=1}^{\infty} S_m(y)\cos(m\pi x) \longleftarrow$$ Fourier cosine series in x with coefficients $S_m(y) = \frac{C_{m0}}{2} + \sum_{n=1}^{\infty} C_{mn}\cos(n\pi y)$ that are themselves Fourier cosine series in y.