HW#2 – Solutions

§1.2

#1. $f(x) = -x$, $-1 \le x \le 1$ $(L=1)$.

Fourier series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

where $a_n = \int_{-1}^{1} \underbrace{\underbrace{-x}_{\text{odd}} \underbrace{\cos(n\pi x)}_{\text{even}}}_{\text{odd}} \, dx = 0$ because the integral is an odd function & the interval is symmetric about the origin

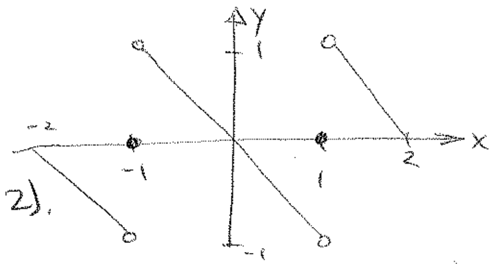
$$b_n = \int_{-1}^{1} -x \sin(n\pi x)\, dx = \underbrace{\cdots}_{\text{int by parts}} = \frac{2(-1)^n}{n\pi} \quad \text{for } n = 1, 2, \ldots$$

$$\text{so } f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) = \frac{2}{\pi}\left(-\sin(\pi x) + \frac{1}{2}\sin(2\pi x) - \frac{1}{3}\sin(3\pi x) + \ldots\right)$$

This series converges to

$$f(x) = \begin{cases} -x & -1 < x < 1 \\ 0 & x = 1 \text{ or } -1 \end{cases}$$

and then extended periodically (w/period 2).



#4. $f(x) = 1 - |x|$ for $-2 \le x \le 2$ $(L=2)$

Fourier series: $f(x) = \frac{a_2}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$.

where $a_n = \frac{1}{2}\int_{-2}^{2} (1-|x|) \cos\left(\frac{n\pi x}{2}\right) dx = \frac{1}{2}\left(\int_{-2}^{0} (1+x)\cos\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{2} (1-x)\cos\left(\frac{n\pi x}{2}\right) dx\right)$

$$= \ldots = -\frac{4((-1)^n - 1)}{n^2\pi^2} = \begin{cases} \frac{8}{n^2\pi^2} & n \text{ odd.} \\ 0 & n \text{ even (including } n=0) \end{cases}$$

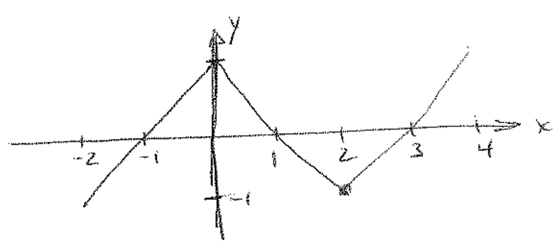
$$b_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\underbrace{(1-|x|)}_{\text{even}} \underbrace{\sin\frac{n\pi x}{2}}_{\text{odd}}}_{\text{odd}} \, dx = 0$$

$$\text{so } f(x) = \frac{8}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \sin\left(\frac{(2k-1)\pi x}{2}\right) = \frac{8}{\pi^2}\left(\sin\frac{\pi x}{2} + \frac{1}{9}\sin\frac{3\pi x}{2} + \frac{1}{25}\sin\frac{5\pi x}{2} + \ldots\right)$$

This series converges to

$f(x) = 1 - |x|$ for $-2 \le x \le 2$

& extended periodically (w/period 4).



#5 $f(x) = \begin{cases} -4 & \text{for } -\pi \le x < 0 \\ 4 & \text{for } 0 \le x \le \pi \end{cases}$ $\quad (L = \pi)$

Fourier series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overset{\text{odd}}{f(x)}\,\overset{\text{even}}{\cos(nx)}}_{\text{odd}}\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -4\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 4\cos(nx)\,dx$

substitution $u = -x$ $\longrightarrow$ $= \frac{4}{\pi}\int_{\pi}^{0} \cos(-nu)(-du) + \frac{4}{\pi}\int_{0}^{\pi}\cos(nx)\,dx = \frac{-4}{\pi}\int_{0}^{\pi}\cos(nx)\,dx + \frac{4}{\pi}\int_{0}^{\pi}\cos(nx)\,dx$

$= 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -4\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 4\sin(nx)\,dx$

$= \frac{-4}{\pi}\int_{\pi}^{0}\sin(-nu)(-du) + \frac{4}{\pi}\int_{0}^{\pi}\sin(nx)\,dx = \frac{8}{\pi}\int_{0}^{\pi}\sin(nx)\,dx$

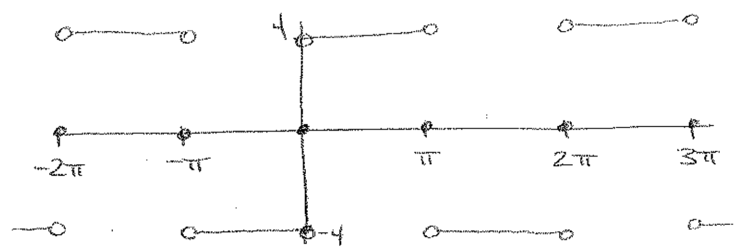
$= \frac{8}{\pi}\left(\frac{-1}{n}\right)\cos(nx)\Big|_{0}^{\pi} = \frac{-8}{n\pi}\left(\cos(n\pi) - \cos(0)\right) = \frac{-8}{n\pi}\left((-1)^n - 1\right)$

$= \begin{cases} \frac{16}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

so $f(x) = \frac{16}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin(\underset{(2k-1)}{\cancel{n}}x) = \frac{16}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots\right)$

This series converges to

$f(x) = \begin{cases} 0 & x = -\pi \\ -4 & -\pi < x < 0 \\ 0 & x = 0 \\ 4 & 0 < x < \pi \\ 0 & x = \pi \end{cases}$

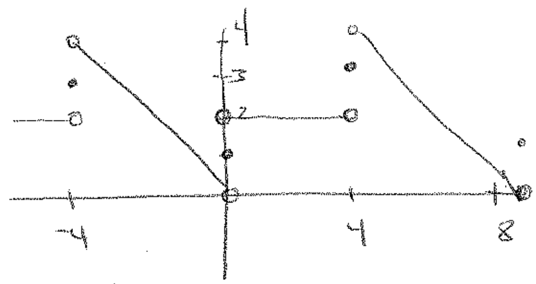


& extended periodically (w/ period 2π).

#9. The Fourier series for $f(x) = \begin{cases} -x & -4 \le x < 0 \\ 2 & 0 \le x \le 4 \end{cases}$

converges to

$f(x) = \begin{cases} 3 & x = -4 \\ -x & -4 < x < 0 \\ 1 & x = 0 \\ 2 & 0 < x < 4 \\ 3 & x = 4 \end{cases}$



& extended periodically (w/ period 8).

#6.

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ -1 & \text{for } 1 < x \le 2 \end{cases} \qquad (L=2)$$

Fourier cosine series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi x}{2}\right)$.

where $a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx - \int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx$

$= \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 = \frac{2}{n\pi}\left(\left(\sin\left(\frac{n\pi}{2}\right) - \cancel{\sin(0)}\right) - \left(\cancel{\sin(n\pi)} - \sin\left(\frac{n\pi}{2}\right)\right)\right)$

$= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$.

Now, when $n=2k$: $\sin\left(\frac{n\pi}{2}\right) = \sin(k\pi) = 0$

$n=4k+1$: $\sin\left(\frac{n\pi}{2}\right) = \sin\left(\frac{(4k+1)\pi}{2}\right) = \sin\left(2k\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$n=4k+3$: $\sin\left(\frac{n\pi}{2}\right) = \sin\left(\frac{(4k+3)\pi}{2}\right) = \sin\left(2k\pi + \frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$.

so $a_n = \begin{cases} \frac{4}{(4k+1)\pi} & n=4k+1 \\ \frac{-4}{(4k+1)\pi} & n=4k+3 \\ 0 & n=2k \end{cases}$. Thus, $f(x) = \frac{4}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} + \ldots\right)$

Fourier sine series: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$

where $b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx = \int_0^1 \sin\frac{n\pi x}{2}dx - \int_1^2 \sin\frac{n\pi x}{2}dx$

$= \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2 = \frac{-2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos 0\right) + \frac{2}{n\pi}\left(\cos(n\pi) - \cos\frac{n\pi}{2}\right)$

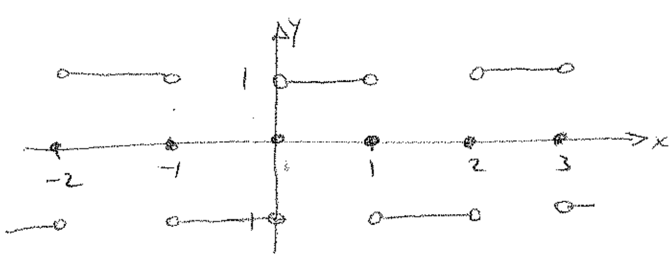
When $n=4k+0$: $\cos(n\pi) - 2\sin\left(\frac{n\pi}{2}\right) + \cos(0) = 0$

$n=4k+1$: $\cos(n\pi) - 2\sin\left(\frac{n\pi}{2}\right) + \cos(0) = 0$

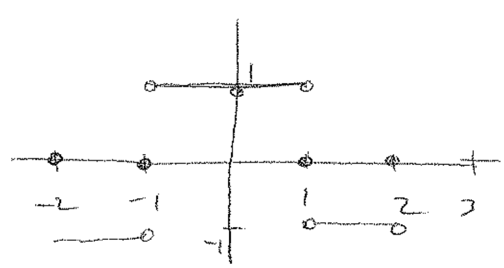
$n=4k+2$: $\cos(n\pi) - 2\sin\left(\frac{n\pi}{2}\right) + \cos(0) = 4$ (this is $n=2, 6, 10, \ldots$)

$n=4k+3$: $\cos(n\pi) - 2\sin\left(\frac{n\pi}{2}\right) + \cos(0) = 0$.

so $b_n = \begin{cases} \frac{8}{n\pi} & \text{if } n=4k+2 \ (n=2,6,10,\ldots) \\ 0 & \text{otherwise.} \end{cases}$ Thus $f(x) = \frac{4}{\pi}\left(\sin(\pi x) + \frac{1}{3}\sin(3\pi x) + \ldots\right)$



Fourier sine series.



Fourier cosine series

#21. $f(x) = \sin(3x)$, $0 \le x \le \pi$, $(L = \pi)$.

Fourier cosine series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

where $a_n = \frac{2}{\pi} \int_0^{\pi} \sin(3x)\cos(nx)\,dx = \dots = \begin{cases} \frac{-12}{(n^2-9)\pi} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$

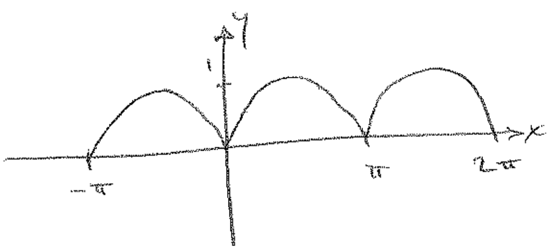
So

$$f(x) = \frac{2}{3\pi} + \frac{12}{5\pi}\cos(2x) - \frac{12}{7\pi}\cos(4x) - \frac{12}{55\pi}\cos(6x) - \frac{12}{91\pi}\cos(8x) - \dots$$

Fourier sine series: $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

where $b_n = \frac{2}{\pi}\int_0^{\pi} \sin(3x)\sin(nx)\,dx = \dots = \begin{cases} 1 & n = 3 \\ 0 & \text{all others.} \end{cases}$

Note that since the given function is itself a Fourier sine series (with $b_n = 0$ except $b_3 = 1$) so this function is its own Fourier sine series.



Fourier cosine series