HW #1 - Solutions

§1.1

#1. $u(x,t) = \cos(\alpha\pi x)\, e^{-\alpha^2\pi^2 t}$

Show: $u_t = u_{xx}$ on $[0,L]$ $(\forall t>0)$

$u_t = -\alpha^2\pi^2 \cos(\alpha\pi x)\, e^{-\alpha^2\pi^2 t}$

$u_x = -\alpha\pi \sin(\alpha\pi x)\, e^{-\alpha^2\pi^2 t}$

$u_{xx} = -(\alpha\pi)^2 \cos(\alpha\pi x)\, e^{-\alpha^2\pi^2 t}$ clearly: $u_t = u_{xx}$.

#4. Let f be a (twice) differentiable function of a single variable.

Show: $u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right)$ solves $u_{tt} = c^2 u_{xx}$ with $u(x,0) = f(x)$.

$u_t = \frac{1}{2}\left(f'(x-ct)(-c) + f'(x+ct)(c)\right) = -\frac{c}{2}\left(f'(x-ct) - f'(x+ct)\right)$

$u_{tt} = -\frac{c}{2}\left(f''(x-ct)(-c) - f''(x+ct)(c)\right) = \frac{c^2}{2}\left(f''(x-ct) + f''(x+ct)\right)$

$u_x = \frac{1}{2}\left(f'(x-ct)(1) + f'(x+ct)(1)\right)$

$u_{xx} = \frac{1}{2}\left(f''(x-ct) + f''(x+ct)\right)$

so $c^2 u_{xx} = \frac{c^2}{2}\left(f''(x-ct) + f''(x+ct)\right) = u_{tt}$.

#7. $\xi = x + at$

$\eta = x + bt$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}^{-1}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$= \frac{1}{b-a}\begin{pmatrix} b & -a \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$= \frac{1}{b-a}\begin{pmatrix} b\xi - a\eta \\ -\xi + \eta \end{pmatrix}$$

$$\therefore\ x = \frac{b\xi - a\eta}{b-a}, \quad y = \frac{-\xi + \eta}{b-a}.$$

#10. $Au_{xx} + Bu_{xt} + Cu_{tt} + H(x,t,u,u_x,u_t) = 0$

Assume $B^2 - 4AC = 0$ and use $a = 0$ and $b = -B/2C$ in the above change of variables: Then $(A + aB + a^2C)V_{\xi\xi} + (2A + (a+b)B + 2abC)V_{\xi\eta} + (A + bB + b^2C)V_{\eta\eta}$

$+ K(\xi,\eta,V,V_\xi,V_\eta) = 0$

Now: $A + aB + a^2C = A$

$2A + (a+b)B + 2abC = 2A + bB = 2A + \frac{-B}{2C}B = \frac{1}{2C}(4AC - B^2) = 0$

$A + bB + b^2C = A + \left(-\frac{B}{2C}\right)B + \left(\frac{B^2}{4C^2}C\right) = A - \frac{B^2}{4C} = 0$

so the general 2nd order PDE becomes:

$$AV_{\xi\xi} + K(\xi,\eta,V,V_\xi,V_\eta) = 0.$$

#11 Assume $B^2-4AC<0$.

The roots of $Ca^2+Ba+A=0$ are $a=\frac{-B\pm\sqrt{B^2-4AC}}{2C} = \underset{p}{\frac{-B}{2C}} \pm i\,\underset{q}{\frac{\sqrt{4AC-B^2}}{2C}}$

Use the change of variables: $\xi = x+pt$, $\eta = qt$

Then $V(\xi,\eta)=u(x,t)$ so that

$$u_t = V_\xi \xi_t + V_\eta \eta_t = V_\xi\cdot(p) + V_\eta\cdot q = pV_\xi + qV_\eta$$

$$u_{tt} = p(pV_\xi+qV_\eta)_\xi + q(pV_\xi+qV_\eta)_\eta = p^2V_{\xi\xi}+2pqV_{\eta\xi}+q^2V_{\eta\eta}$$

$$u_x = V_\xi\xi_x+V_\eta\eta_x = V_\xi$$

$$u_{xx} = (V_\xi)_\xi\xi_x+(V_\xi)_\eta\eta_x = V_{\xi\xi}, \quad u_{xt} = (V_\xi)_\xi\xi_t+(V_\xi)_\eta\eta_t = pV_{\xi\xi}+qV_{\xi\eta}$$

and the PDE takes the form:

$$0 = Au_{xx}+Bu_{xt}+Cu_{tt}+H(x,t,u,u_x,u_t)$$

$$= AV_{\xi\xi}+B(pV_{\xi\xi}+qV_{\xi\eta})+C(p^2V_{\xi\xi}+2pqV_{\eta\xi}+q^2V_{\eta\eta})+K(\xi,\eta,V,V_\xi,V_\eta)$$

$$=(A+pB+p^2C)V_{\xi\xi}+(qB+2pqC)V_{\xi\eta}+q^2CV_{\eta\eta}+K(\xi,\eta,V,V_\xi,V_\eta)$$

Now:

$$A+pB+p^2C = A+\left(-\frac{B}{2C}\right)B+\left(-\frac{B}{2C}\right)^2C = A-\frac{B^2}{2C}+\frac{B^2}{4C} = A-\frac{B^2}{4C}$$

$$qB+2pqC = q(B+2pC) = q\left(B+2\left(-\frac{B}{2C}\right)C\right) = q(B-B)=0.$$

$$q^2C = \left(\frac{\sqrt{4AC-B^2}}{2C}\right)^2C = \frac{4AC-B^2}{4C} = A-\frac{B^2}{4C}$$

so that the transformed PDE becomes:

$$\left(A-\frac{B^2}{4C}\right)V_{\xi\xi}+\left(A-\frac{B^2}{4C}\right)V_{\eta\eta}+K(\xi,\eta,V,V_\xi,V_\eta)=0.$$

Divide by $A-\frac{B^2}{4C}$ ($\neq 0$) to finish.

#14. $2u_{xx}+u_{xt}-4u_{tt}+x+t=0.$ $\quad(A=2,\ B=1,\ C=-4)$ $\quad a=\frac{-1+\sqrt{33}}{-8}$

$B^2-4AC = (1)^2-4(2)(-4)=33>0$ so this PDE is hyperbolic. $\quad b=\frac{-1-\sqrt{33}}{-8}$

Use the change of variables with $\xi = x+\left(\frac{-1\pm\sqrt{33}}{-8}\right)t$, $\eta = x-\left(\frac{-1+\sqrt{33}}{-8}\right)t$

The result is: $(2A+(a+b)B+2abC)V_{\xi\eta}+\underbrace{\frac{b\xi-a\eta}{b-a}}_{x}+\underbrace{\frac{\eta-\xi}{b-a}}_{t}=0$

$$0=\left(2(2)+\left(\frac{-1+\sqrt{33}}{-8}+\frac{-1-\sqrt{33}}{-8}\right)(1)+2\left(\frac{-1+\sqrt{33}}{-8}\right)\left(\frac{-1-\sqrt{33}}{-8}\right)(-4)\right)V_{\xi\eta}+\frac{\frac{-1-\sqrt{33}}{-8}\xi-\frac{-1+\sqrt{33}}{-8}\eta}{\frac{-1-\sqrt{33}}{-8}-\frac{-1+\sqrt{33}}{-8}}+\frac{\eta-\xi}{\frac{-1-\sqrt{33}}{-8}-\frac{-1+\sqrt{33}}{-8}}$$

$$=\frac{17}{4}V_{\xi\eta}+\left(\frac{1}{2}+\frac{1}{\sqrt{33}}\right)\xi+\left(\frac{1}{2}-\frac{1}{\sqrt{33}}\right)\eta$$