# Math 521 - Exam 2
## Solutions

#1. $\nabla^2 u = 0 \qquad 1 \le r \le 2, \quad -\pi < \theta \le \pi$

$$\left.\begin{array}{l} u(1,\theta) = \sin(3\theta) \\ \frac{\partial u}{\partial r}(2,\theta) = \cos(2\theta) \end{array}\right\} \quad -\pi < \pi \le \pi$$

The general solution is a superposition of the harmonic functions in polar coord:

$$u(r,\theta) = \frac{a_0}{2} + c_0 \ln r + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) + \sum_{n=1}^{\infty} r^{-n}\left(c_n \cos(n\theta) + d_n \sin(n\theta)\right)$$

$$= \frac{a_0}{2} + c_0 \ln r + \sum_{n=1}^{\infty} \left(a_n r^n + c_n r^{-n}\right) \cos(n\theta) + \sum_{n=1}^{\infty} \left(b_n r^n + d_n r^{-n}\right) \sin(n\theta)$$

Then $u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n + c_n) \cos(n\theta) + \sum_{n=1}^{\infty} (b_n + d_n) \sin(n\theta) = \sin(3\theta)$

provided $\frac{a_0}{2} = 0, \quad a_n + c_n = 0 \ (n \ge 1), \quad b_3 + d_3 = 1, \quad b_n + d_n = 0 \ (n \ge 1, n \ne 3)$.

Differentiating the gen'l sol'n wrt $r$:

$$\frac{\partial u}{\partial r}(r,\theta) = \frac{c_0}{r} + \sum_{n=1}^{\infty} n\left(a_n r^{n-1} - c_n r^{-n-1}\right) \cos(n\theta) + \sum_{n=1}^{\infty} n\left(b_n r^{n-1} - d_n r^{-n-1}\right) \sin(n\theta)$$

Then $\frac{\partial u}{\partial r}(2,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} n\left(2^{n-1} a_n - 2^{-n-1} c_n\right) \cos(n\theta) + \sum_{n=1}^{\infty} n\left(2^{n-1} b_n - 2^{-n-1} d_n\right) \sin(n\theta) = \cos(2\theta)$

provided $\frac{c_0}{2} = 0, \ 2\left(2a_2 - 2^{-3} c_2\right) = 1, \ 2^{n-1} a_n - 2^{-n-1} c_n = 0 \ (n \ge 1, n \ne 2), \quad 2^{n-1} b_n - 2^{-n-1} d_n = 0 \ (n \ge 1)$

we see that $a_0 = 0, c_0 = 0$

To determine all coefficients

$$\left.\begin{array}{l} a_2 + c_2 = 0 \\ 2a_2 - \frac{1}{8} c_2 = 1/2 \end{array}\right\} \Rightarrow \frac{17}{8} a_2 = 1/2 \Rightarrow a_2 = \frac{4}{17}, \quad c_2 = -a_2 = -\frac{4}{17}$$

$$\left.\begin{array}{l} a_n + c_n = 0 \\ 2^{n-1} a_n - 2^{-n-1} c_n = 0 \end{array}\right\} \Rightarrow a_n = c_n = 0 \quad \text{for all } n \ge 1, n \ne 2.$$

$$\left.\begin{array}{l} b_3 + d_3 = 1 \\ 4b_3 - \frac{1}{16} d_3 = 0 \end{array}\right\} \Rightarrow \frac{65}{16} b_3 = \frac{1}{16} \Rightarrow b_3 = \frac{1}{65}, \quad d_3 = 1 - b_3 = \frac{64}{65}$$

$$\left.\begin{array}{l} b_n + d_n = 0 \\ 2^{n-1} b_n - 2^{-n-1} d_n = 0 \end{array}\right\} \Rightarrow b_n = d_n = 0 \quad \text{for all } n \ge 1, n \ne 3.$$

The final solution is $u(r,\theta) = \left(\frac{8}{17} r^2 + \frac{-8}{17} r^{-2}\right) \cos(2\theta) + \left(\frac{64}{65} r^{-3} + \frac{1}{65} r^3\right) \sin(3\theta)$

$$= \frac{8}{17}\left(r^2 - \frac{1}{r^2}\right) \cos(2\theta) + \frac{1}{65}\left(r^3 + \frac{64}{r^3}\right) \sin(3\theta).$$

#2. $\nabla^2 u = 0 \quad 0<x<\pi,\ 0<y<\pi$

$u(0,y)=0,\ u(\pi,y)=1 \quad 0<y<\pi$

$u_y(x,0)=\cos(3x),\ u_y(x,\pi)=0 \quad 0<x<\pi$

We separate this into 2 problems, each with non-zero data on only 1 side.

Thus, $u_1$ solves $\nabla^2 u=0 \quad 0<x<\pi,\ 0<y<\pi$

$u(0,y)=u(\pi,y)=0 \quad 0<y<\pi$

$u_y(x,0)=\cos(3x),\ u_y(x,\pi)=0 \quad 0<x<\pi$

and $u_2$ solves $\nabla^2 u=0 \quad 0<x<\pi,\ 0<y<\pi$

$u(0,y)=0,\ u(\pi,y)=1 \quad 0<y<\pi$

$u_y(x,0)=u_y(x,\pi)=0 \quad 0<x<\pi$

First, we find $u_1$: By separation of variables we know we'll find:

$X''-\lambda X=0 \qquad Y''+\lambda Y=0$

$X(0)=X(\pi)=0 \qquad Y'(\pi)=0$

$\Downarrow$

$\lambda_n=n^2\ (n\geq 1)$

$X_n(x)=\sin(nx)$

Then the sol'n to the $Y$ equation is

$$Y_n(y)=a_n e^{ny}+b_n e^{-ny}$$

To satisfy the boundary condition $Y'(\pi)=0$:

$$Y_n'(y)=na_ne^{ny}-nb_ne^{ny}$$

$$Y_n'(\pi)=na_ne^{n\pi}-nb_ne^{-n\pi}=0 \Rightarrow b_n=e^{2n\pi}a_n$$

$$\therefore Y_n(y)=e^{ny}+e^{2n\pi}e^{-ny}=e^{ny}+e^{2n\pi-ny}.$$

So $u_1(x,y)=\sum_{n=1}^{\infty} a_n\sin(nx)\left(e^{ny}+e^{2n\pi-ny}\right)$

Then $\frac{\partial u_1}{\partial y}=\sum_{n=1}^{\infty} a_n\sin(nx)\left(ne^{ny}-ne^{2n\pi-ny}\right)$

$$\frac{\partial u_1}{\partial y}(x,0)=\sum_{n=1}^{\infty} a_n\left(n-ne^{2n\pi}\right)\sin(nx)=\cos(3x).$$

so $a_n(n-ne^{2n\pi})=\frac{2}{\pi}\int_0^\pi \cos(3x)\sin(nx)\,dx$

Note that, by Maple, $\int_0^\pi \cos(3x)\sin(nx)\,dx=\frac{(1+(-1)^n)n}{n^2-9}=\begin{cases}\frac{2n}{n^2-9} & n \text{ even}\\ 0 & n \text{ odd}\end{cases}$ (even for $n=3$).

so $u_1(x,y)=\sum_{n=1}^{\infty} a_{2n}\sin(2nx)\left(e^{2ny}+e^{4n\pi-2ny}\right)=\sum_{n=1}^{\infty}\frac{2}{\pi}\frac{4n}{(4n^2-9)}\sin(2nx)\frac{e^{2ny}+e^{4n\pi-2ny}}{n(1-e^{4n\pi})}$

To find $u_2$ we again look to separation of variables.

$$X'' - \lambda X = 0 \qquad Y'' + \lambda Y = 0$$
$$X(0) = 0 \qquad \underbrace{Y'(0) = Y'(\pi) = 0}$$

$$\Downarrow$$

$$\lambda_n = n^2 \quad (n \geq 0)$$
$$Y_n(y) = \cos(ny) \qquad (\text{including } Y_0(y) = 1)$$

Then $X_n(x) = a_n e^{nx} + b_n e^{-nx}$ (for $n \geq 1$) and $X_0(x) = a_0 x + b_0$

To satisfy the IC for $X$:

$n = 0$: $X_0(0) = b_0 = 0 \implies X_0(x) = x$

$n \geq 1$: $X_n(0) = a_n + b_n = 0 \implies X_n(x) = e^{nx} - e^{-nx}$

$(b_n = -a_n)$

So $u_2(x,y) = \frac{a_0}{2} x + \sum_{n=1}^{\infty} a_n (e^{nx} - e^{-nx}) \cos(ny)$

Then $u_2(\pi, y) = \frac{a_0 \pi}{2} + \sum_{n=1}^{\infty} a_n (e^{n\pi} - e^{-n\pi}) \cos(ny) = 1$

when $\frac{a_0 \pi}{2} = 1 \implies a_0 = \frac{2}{\pi}$

and $a_n (e^{n\pi} - e^{-n\pi}) = 0 \quad (n \geq 1) \implies a_n = 0 \ (n \geq 1)$.

So $u_2(x,y) = \frac{x}{\pi}$.

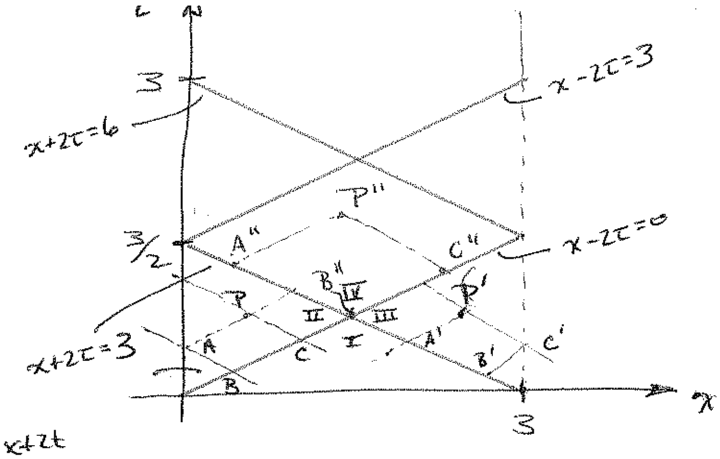
To combine these we have

$$u(x,y) = u_1(x,y) + u_2(x,y)$$
$$= \frac{x}{\pi} + \sum_{n=1}^{\infty} \frac{8}{\pi(4n^2-9)} \sin(2nx) \frac{e^{2ny} + e^{4n\pi - 2ny}}{1 - e^{4n\pi}}$$

#3. $u_{tt} = 4u_{xx}$ $\quad 0<x<3,\ t>0$

$$\left.\begin{array}{l} u(x,0)=\sin(\pi x) \\ u_t(x,0)=0 \end{array}\right\} \ 0<x<3$$

$u(0,t)=u(3,t)=t^3 \quad t>0$




In Region I we use d'Alembert's formula for the solution:

$$u(x,t)=\frac12\left(\varphi(x+2t)+\varphi(x-2t)\right)+\frac{1}{2\cdot 2}\int_{x-2t}^{x+2t}\psi(s)\,ds = \frac12\left(\sin(\pi(x+2t))+\sin(\pi(x-2t))\right)$$

$$=\frac12\left(\sin\pi x\cos 2\pi t+\cancel{\cos(\pi x)\sin(2\pi t)}+\sin(\pi x)\cos(2\pi t)-\cancel{\cos(\pi x)\sin(2\pi t)}\right)$$

$$=\sin(\pi x)\cos(2\pi t).$$

In Region II, let P be a point $(x,y)$ then $u(P)=u(A)+u(C)-u(B)$

where A is on the intersection of $x-2\tau=x-2t$ & $x=0$, thus $\tau=\frac{x-2t}{-2}=t-\frac{x}{2}$.

B is on the intersection of $x-2\tau=0$ and $x+2\tau=2t-x$: $\begin{array}{l}2x=2t-x\\4\tau=2t-x\end{array} \Rightarrow \begin{array}{l}x=\frac12(2t-x)\\ \tau=\frac14(2t-x)\end{array}$

C is on the intersection of $x-2\tau=0$ and $x+2\tau=x+2t$: $\begin{array}{l}2x=x+2t\\4\tau=x+2t\end{array} \Rightarrow \begin{array}{l}x=\frac12(2t+x)\\ \tau=\frac14(2t+x)\end{array}$

~~B is on the intersection of~~

Thus $u(A)=u\left(0,\frac12(2t-x)\right)=\left(\frac12(2t-x)\right)^3=\frac18(2t-x)^3$

$u(B)=u\left(\frac12(2t-x),\frac14(2t-x)\right)=\sin\left(\pi\cdot\frac12(2t-x)\right)\cos\left(2\pi\cdot\frac14(2t-x)\right)$

$=\sin\left(\pi t-\frac{\pi x}{2}\right)\cos\left(\pi t-\frac{\pi x}{2}\right)$

$u(C)=u\left(\frac12(2t+x),\frac14(2t+x)\right)=\sin\left(\pi\cdot\frac12(2t+x)\right)\cos\left(2\pi\cdot\frac14(2t+x)\right)$

$=\sin\left(\pi t+\frac{\pi x}{2}\right)\cos\left(\pi t+\frac{\pi x}{2}\right)$

and so $u(P)=u(A)+u(C)-u(B)$

$=\frac18(2t-x)^3+\sin\left(\pi t+\frac{\pi x}{2}\right)\cos\left(\pi t+\frac{\pi x}{2}\right)-\sin\left(\pi t+\frac{\pi x}{2}\right)\cos\left(\pi t-\frac{\pi x}{2}\right)$

$=\frac18(2t-x)^3+\frac12\sin(2\pi t+\pi x)-\frac12\sin(2\pi t-\pi x)$

$=\frac18(2t-x)^3+\sin(\pi x)\cos(2\pi t).$

Extra Credit

Region III: Let $P' = (x,t)$ then the other 3 vertices are:

$A'$: $x - 2\tau = x - 2t$
$x + 2\tau = 3$
$2x = 3 + x - 2t$
$4\tau = 3 - (x-2t)$

$$\begin{cases} x = \frac{1}{2}(3 + x - 2t) \\ \tau = \frac{1}{4}(3 - (x-2t)) \end{cases}$$

$B'$: $x + 2\tau = 3$
$x - 2\tau = 6 - x - 2t$
$2x = 9 - (x+2t)$
$4\tau = -3 + (x+2t)$

$$\begin{cases} x = \frac{9}{2} - \frac{1}{2}(x+2t) \\ \tau = -\frac{3}{4} + \frac{1}{4}(x+2t) \end{cases}$$

$C'$: $x + 2\tau = x + 2t$
$x = 3$
$2\tau = x + 2t - 3$

$$\begin{cases} x = 3 \\ \tau = \frac{1}{2}(x+2t-3) \end{cases}$$

so
$$\begin{aligned} u(A') &= u\left(\tfrac{1}{2}(3+x-2t), \tfrac{1}{4}(3-(x-2t))\right) \\ &= \sin\left(\tfrac{3\pi}{2} + \tfrac{\pi}{2}(x-2t)\right)\cos\left(\tfrac{3\pi}{2} - \tfrac{\pi}{2}(x-2t)\right) \\ &= \left(-\cos\left(\tfrac{\pi}{2}(x-2t)\right)\right)\left(-\sin\left(\tfrac{\pi}{2}(x-2t)\right)\right) \\ &= \tfrac{1}{2}\sin(\pi x - 2\pi t) \end{aligned}$$

$$\begin{aligned} u(B') &= u\left(\tfrac{9}{2} - \tfrac{1}{2}(x+2t), -\tfrac{3}{4} + \tfrac{1}{4}(x+2t)\right) \\ &= \sin\left(\tfrac{9\pi}{2} - \tfrac{\pi}{2}(x+2t)\right)\cos\left(-\tfrac{3\pi}{2} + \tfrac{\pi}{2}(x+2t)\right) \\ &= \cos\left(\tfrac{\pi}{2}(x+2t)\right)\left(-\sin\left(\tfrac{\pi}{2}(x+2t)\right)\right) \\ &= -\tfrac{1}{2}\sin(\pi x + 2\pi t) \end{aligned}$$

$$u(C') = u\left(3, \tfrac{1}{2}(x+2t-3)\right) = \left(\tfrac{1}{2}(x+2t-3)\right)^3 = \tfrac{1}{8}(x+2t-3)^3$$

Finally,
$$\begin{aligned} u(x,t) = u(P') &= u(A') + u(C') - u(B') \\ &= \tfrac{1}{8}(x+2t-3)^3 + \tfrac{1}{2}\sin(\pi x - 2\pi t) - \left(-\tfrac{1}{2}\sin(\pi x + 2\pi t)\right) \\ &= \tfrac{1}{8}(x+2t-3)^3 + \tfrac{1}{2}\left(\sin(\pi x - 2\pi t) + \sin(\pi x + 2\pi t)\right) \\ &= \tfrac{1}{8}(x+2t-3)^3 + \tfrac{1}{2}\Big(\sin(\pi x)\cos(2\pi t) - \cancel{\cos(\pi x)\sin(2\pi t)} \\ &\qquad + \sin(\pi x)\cos(2\pi t) + \cancel{\cos(\pi x)\sin(2\pi t)}\Big) \\ &= \tfrac{1}{8}(x+2t-3)^3 + \sin(\pi x)\cos(2\pi t) \end{aligned}$$

## Region IV (Extra Credit)

Let $P''$ be a point $(x,t)$ in region IV. Then $u(P'') = u(A'') + u(C'') - u(B'')$.
All that remains is to determine $A''$, $B''$, and $C''$ in terms of $x$ & $t$.

$$A'': \quad \begin{aligned} x - 2\tau &= x - 2t \\ x + 2\tau &= 3 \\ \hline 2x &= 3 + x - 2t \\ 4\tau &= 3 - (x-2t) \end{aligned} \qquad \therefore \quad \begin{aligned} x &= \tfrac{3}{2} + \tfrac{1}{2}(x-2t) \\ \tau &= \tfrac{3}{4} - \tfrac{1}{4}(x-2t) \end{aligned}$$

$$B'': \quad \begin{aligned} x + 2\tau &= 3 \\ x - 2\tau &= 0 \\ \hline 2x &= 3 \\ 4\tau &= 3 \end{aligned} \qquad \therefore \quad \begin{aligned} x &= \tfrac{3}{2} \\ \tau &= \tfrac{3}{4}. \end{aligned}$$

$$C'': \quad \begin{aligned} x + 2\tau &= x + 2t \\ x - 2\tau &= 0 \\ \hline 2x &= x + 2t \\ 4\tau &= x + 2t \end{aligned} \qquad \therefore \quad \begin{aligned} x &= \tfrac{1}{2}(x+2t) \\ \tau &= \tfrac{1}{4}(x+2t) \end{aligned}$$

Now, we find $u(A'')$ from region II, $u(C'')$ from region III, and $u(B'')$ can be found from region I, II, or III:

$$u(A'') = u\left(\tfrac{3}{2} + \tfrac{1}{2}(x-2t), \tfrac{3}{4} - \tfrac{1}{4}(x-2t)\right) = \frac{1}{8}\left(\left(2\left(\tfrac{3}{4} - \tfrac{1}{4}(x-2t)\right) - \left(\tfrac{3}{2} + \tfrac{1}{2}(x-2t)\right)\right)^3 + \sin\left(\pi\left(\tfrac{3}{2} + \tfrac{1}{2}(x-2t)\right)\right)\cos\left(2\pi\left(\tfrac{3}{4} - \tfrac{1}{4}(x-2t)\right)\right)\right)$$

$$\begin{aligned}
&= \frac{1}{8}(2t - x)^3 + \sin\left(\frac{3\pi}{2} + \frac{\pi}{2}(x-2t)\right)\cos\left(\frac{3\pi}{2} - \frac{\pi}{2}(x-2t)\right) \\
&= \frac{1}{8}(2t-x)^3 - \cos\left(\frac{\pi}{2}(x-2t)\right)\left(-\sin\left(\frac{\pi}{2}(x-2t)\right)\right) \\
&= \frac{1}{8}(2t-x)^3 + \sin\left(\frac{\pi}{2}(x-2t)\right)\cos\left(\frac{\pi}{2}(x-2t)\right)
\end{aligned}$$

$$\begin{aligned}
u(C'') &= \frac{1}{8}\left(\frac{1}{2}(x+2t) + \frac{1}{2}(x+2t) - 3\right)^3 + \sin\left(\frac{\pi}{2}(x+2t)\right)\cos\left(\frac{\pi}{2}(x+2t)\right) \\
&= \frac{1}{8}(x + 2t - 3)^3 + \sin\left(\frac{\pi}{2}(x+2t)\right)\cos\left(\frac{\pi}{2}(x+2t)\right)
\end{aligned}$$

$$u(B'') = \sin\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{2}\right) = (-1)\,0 = 0.$$

$\left\{ \text{Note that at } B'' = \left(\tfrac{3}{2}, \tfrac{3}{4}\right) \;\; 2t - x = \tfrac{3}{2} - \tfrac{3}{2} = 0 \text{ and } x + 2t - 3 = \tfrac{3}{2} + \tfrac{3}{2} - 3 = 0 \right\}$

To conclude:

$$\begin{aligned}
u(x,t) = u(A'') + u(C'') - u(B'') &= \frac{1}{8}(2t-x)^3 + \sin\left(\frac{\pi}{2}(x-2t)\right)\cos\left(\frac{\pi}{2}(x-2t)\right) \\
&\quad + \frac{1}{8}(x+2t-3)^3 + \sin\left(\frac{\pi}{2}(x+2t)\right)\cos\left(\frac{\pi}{2}(x+2t)\right) \\
&= \frac{1}{8}(2t-x)^3 + \frac{1}{8}(x+2t-3)^3 + \frac{1}{2}\sin(\pi(x-2t)) + \frac{1}{2}\sin(\pi(x+2t)) \\
&= \frac{1}{8}(2t-x)^3 + \frac{1}{8}(x+2t-3)^3 + \frac{1}{2}\left(\sin(\pi x)\cos(2\pi t) \cancel{- \cos(\pi x)\sin(2\pi t)} + \sin(\pi x)\cos(2\pi t) \cancel{+ \cos(\pi x)\sin(2\pi t)}\right) \\
&= \frac{1}{8}(2t-x)^3 + \frac{1}{8}(x+2t-3)^3 + \sin(\pi x)\cos(2\pi t).
\end{aligned}$$

7.

#4. $u_{tt} = u_{xx} + u_{yy}$ $\quad 0 < x < 1,\ 0 < y < \pi,\ t > 0$

$u(x,0,t) = u(x,\pi,t) = 0 \quad 0 < x < 1,\ t > 0$

$u_x(0,y,t) = u_x(1,y,t) = 0 \quad 0 < y < 1,\ t > 0$

$$\left.\begin{array}{l} u(x,y,0) = y\cos\left(\frac{\pi x}{2}\right) \\ u_t(x,y,0) = x+y \end{array}\right\} \quad 0 < x < 1,\ 0 < y < \pi.$$

Looking $u(x,y,t) = X(x)Y(y)T(t)$ by separation of variables leads to

$$\frac{XYT''}{XYT} = \frac{X''YT + XY''T}{XYT} \quad \text{so} \quad \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$$

This leads to considering $X'' + \lambda X = 0,\quad Y'' + \mu Y = 0,\quad T'' + (\lambda+\mu)T = 0$

The boundary conditions add: $\underbrace{X'(0) = X'(1) = 0}_{\Downarrow}$ $\quad \underbrace{Y(0) = Y(\pi) = 0}_{\Downarrow}$

$\lambda_n = (n\pi)^2$ for $n \geq 0$ $\qquad \mu_m = m^2$ for $m \geq 1$.

$X_n(x) = \cos(n\pi x)$ $\qquad Y_m(y) = \sin(my)$

Then the T equation becomes: $T'' + \left((n\pi)^2 + m^2\right)T = 0$.

Its solutions are $T_{mn}(t) = a_{mn}\sin\left(\sqrt{n^2\pi^2+m^2}\,t\right) + b_{mn}\cos\left(\sqrt{n^2\pi^2+m^2}\,t\right)$

We put these together to find the general solution in the form

$$u(x,y,t) = \frac{1}{2}\sum_{m=1}^{\infty}\left(a_{om}\sin(mt) + b_{om}\cos(mt)\right)\sin(my)$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(a_{nm}\sin\left(\sqrt{n^2\pi^2+m^2}\,t\right) + b_{nm}\cos\left(\sqrt{n^2\pi^2+m^2}\,t\right)\right)\cos(n\pi x)\sin(my)$$

Then $u(x,y,0) = \frac{1}{2}\sum_{m=1}^{\infty} b_{om}\sin(my) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\cos(n\pi x)\sin(my) = y\cos\left(\frac{\pi x}{2}\right)$

so $b_{nm} = \frac{2}{1}\frac{2}{\pi}\int_0^1\int_0^{\pi} y\cos\left(\frac{\pi x}{2}\right)\cos(n\pi x)\sin(my)\,dy\,dx$

$$= \frac{4}{\pi}\int_0^1 \cos\left(\frac{\pi x}{2}\right)\cos(n\pi x)\,dx \int_0^{\pi} y\sin(my)\,dy = \frac{8}{\pi}\frac{(-1)^{m+n}}{m(4n^2-1)}$$

Maple.

Next, differentiating the solution wrt t:

$$u_t(x,y,t) = \frac{1}{2}\sum_{m=1}^{\infty} m\left(a_{0m}\cos(mt) - b_{0m}\sin(mt)\right)\sin(my)$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sqrt{n^2\pi^2+m^2}\left(a_{nm}\cos\left(\sqrt{n^2\pi^2+m^2}\,t\right) - b_{nm}\sin\left(\sqrt{n^2\pi^2+m^2}\,t\right)\right)\cos(n\pi x)\sin(my)$$

so that

$$u_t(x,y,0) = \frac{1}{2}\sum_{m=1}^{\infty} m\, a_{0m}\sin(my) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sqrt{n^2\pi^2+m^2}\; a_{nm}\cos(n\pi x)\sin(my) = x+y$$

This will be satisfied when

$$a_{nm} = \frac{2}{1}\,\frac{2}{\pi}\int_0^1\int_0^{\pi}(x+y)\cos(n\pi x)\sin(my)\,dy\,dx \underset{\text{Maple}}{=} \begin{cases} \dfrac{2\left(1-(1+2\pi)(-1)^m\right)}{\pi m^2} & n=0. \\[2ex] \dfrac{-4\left(1-(-1)^n\right)\left(1-(-1)^m\right)}{\pi^3 m n^2\sqrt{n^2\pi^2+m^2}} & n\geq 1. \end{cases}$$

Note that $a_{nm}=0$ for $n\geq 1$ whenever either $n$ is even or $m$ is even. The only non-zero coefficients are $n=0$ (all $m$) and when both $m$ & $n$ are odd.